Unit 1: Probability

1.1: Experimental and Theoretical Probability

Experimental Probability: - probability that came from a simulation such as tossing dice, coins … etc.

Binomial Distribution: - a sample space where there is only two outcomes (favorable, non-favorable).

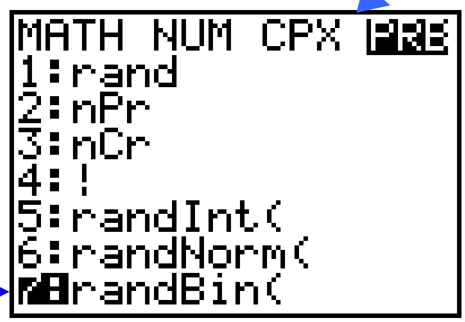
randBin (Random Binomial): generates and displays a random integer from a specified binomial distribution.

randBin (number of trials, Theoretical Probability of favorable outcome, number of simulations)

Always set to 1 (number of trials)

To access randBin:

1. Press **MATH**
2. Use  to access PRB
3. Select Option 7



Theoretical Probability: - probability from calculations by using sample space or other formulas.

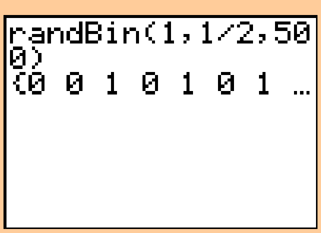
Example 1: Find the probability of obtaining tails when a coin is tossed 500 times theoretically and experimentally.

Theoretical Probability

Sample Space of Tossing a Coin = (Head), (Tail) $P\text{(Tail)} = \frac{1}{2}$

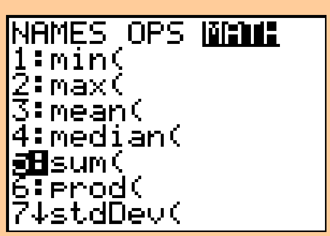
Experimental Probability using Graphing Calculator

1. **randBin (1, ½, 500)**

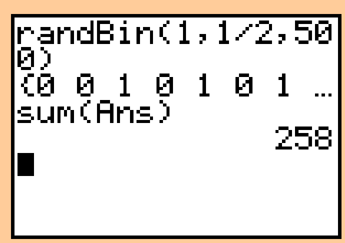


2. Summing all trails of favorable outcome. **sum (Ans)**

2nd **LIST STAT**



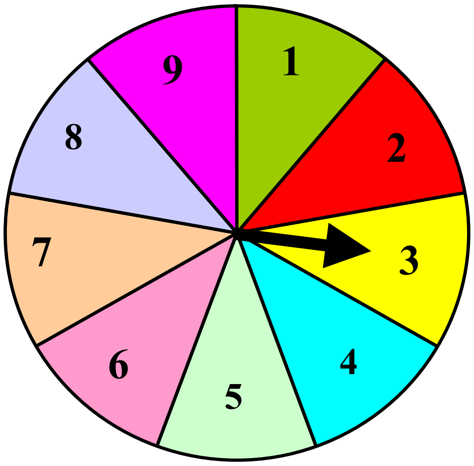
2nd **ANS (−)**



Experimental P (Tail) = $\frac{258}{500} = 0.516 = \frac{129}{250}$

(Your answer might be different depending on when you last RESET your calculator.)

Example 2: Find the theoretical and experimental probability of **NOT** landing on a 5 using the spinner below if it was spun 100 times.



Theoretical Probability

$P(5) = \frac{1}{9}$

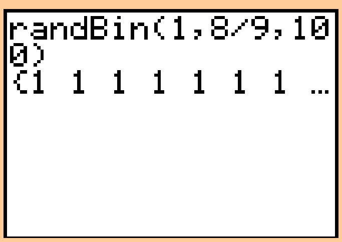
$P(\bar{5}) = 1 - \frac{1}{9}$

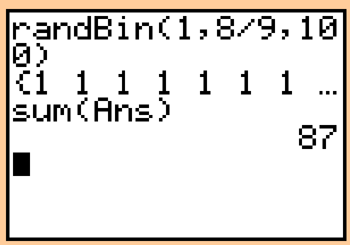
P (NOT 5)

$\boldsymbol{P(\bar{5}) = \frac{8}{9}}$

Experimental Probability using Graphing Calculator

1. **randBin (1, 8/9, 100)**
2. Summing all simulations of favorable outcome.





Experimental $\boldsymbol{P(\bar{5}) = \frac{87}{100} = 0.87}$

Example 3: What is the theoretical probability of the following from a standard deck of 52 cards?

a. Any ace.

(There are 4 Aces in a deck of cards.)

$P(\text{Ace}) = \frac{4}{52}$

$\boldsymbol{P(\text{Ace}) = \frac{1}{13}}$

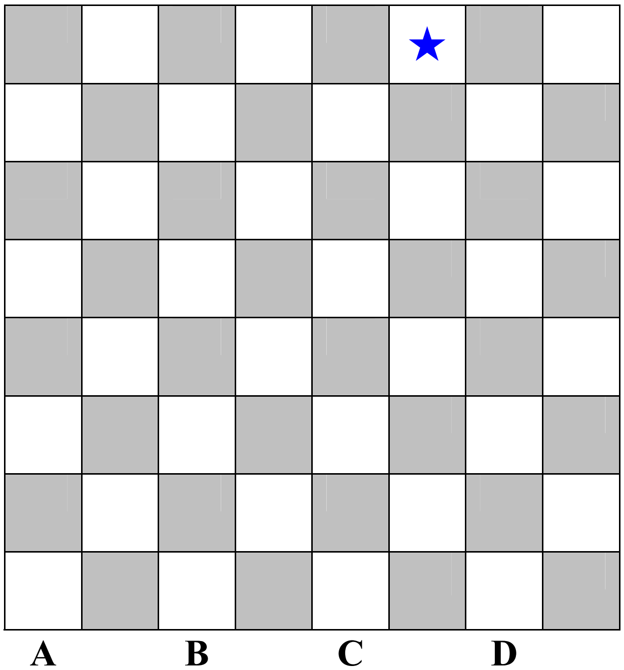
b. Any number cards.

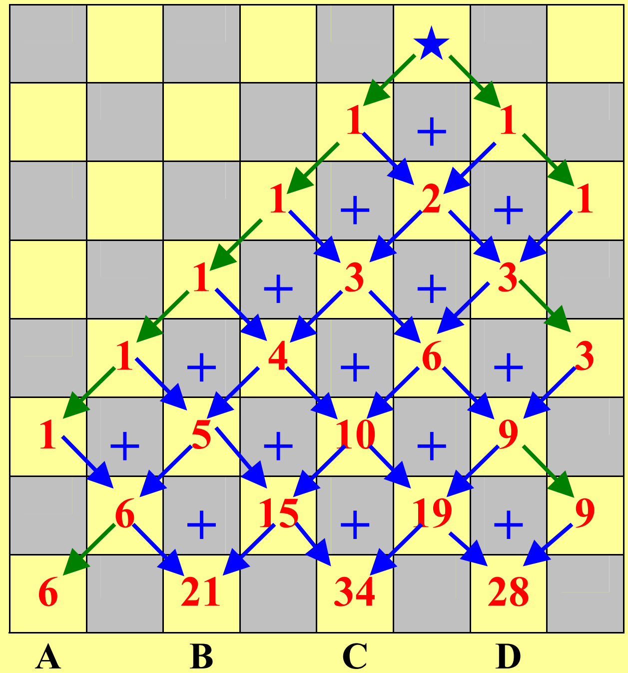
There are 36 Number cards in a deck.
({2,3,4,5,6,7,8,9,10} × 4 suits)

$P(\text{Number Cards}) = \frac{36}{52}$

$\boldsymbol{P(\text{Number Cards}) = \frac{9}{13}}$

Example 4: Find the number of ways to reach the other side of the checkerboard if you can only move forward diagonally from ★.

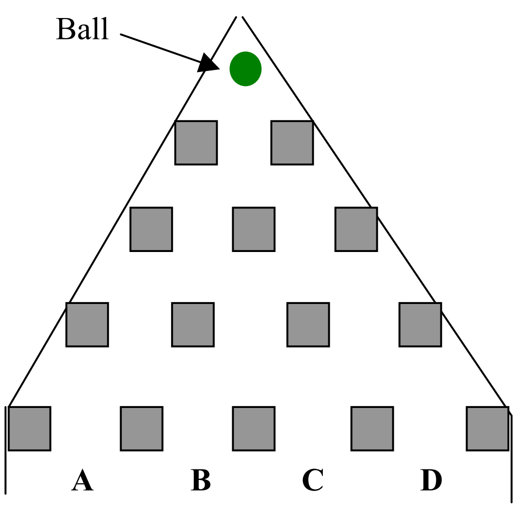


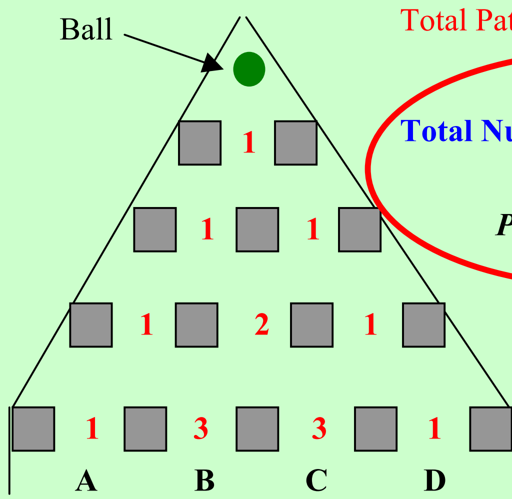


Total Pathways = 6 + 21 + 34 + 28

Total Number of Pathways = 89

Example 5: Determine the number of ways a pinball will pass through set up below. What is the probability that it will exit at B?





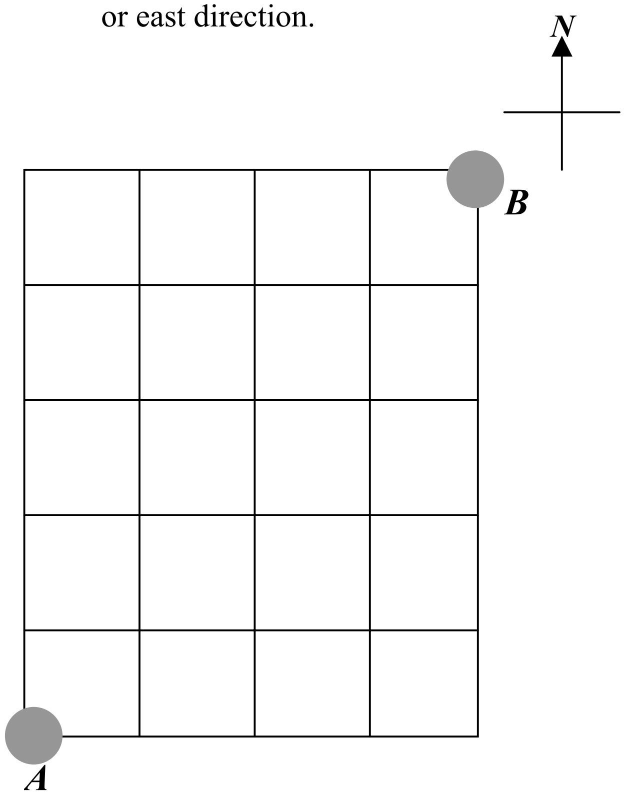
Total Pathways = 1 + 3 + 3 + 1

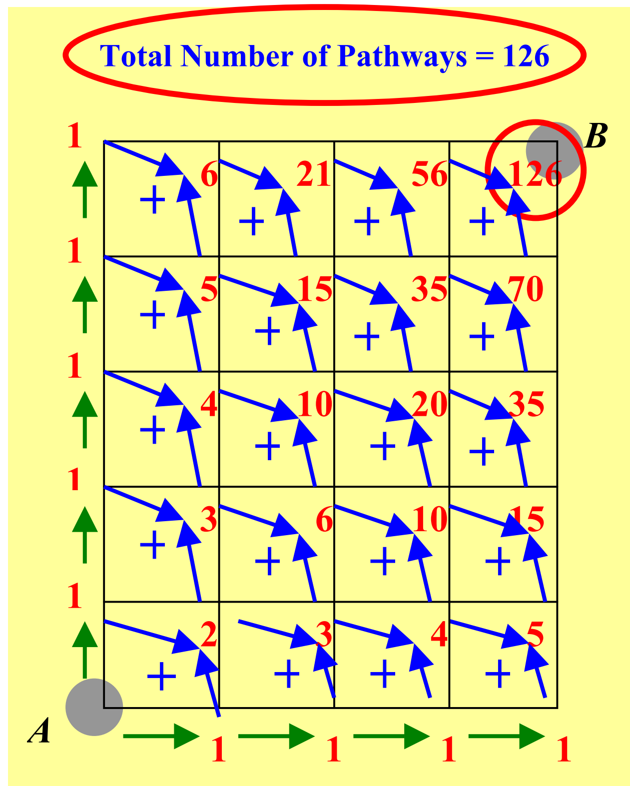
Total Number of Pathways = 8

$P\,(\text{exit } B) = \dfrac{3}{8}$

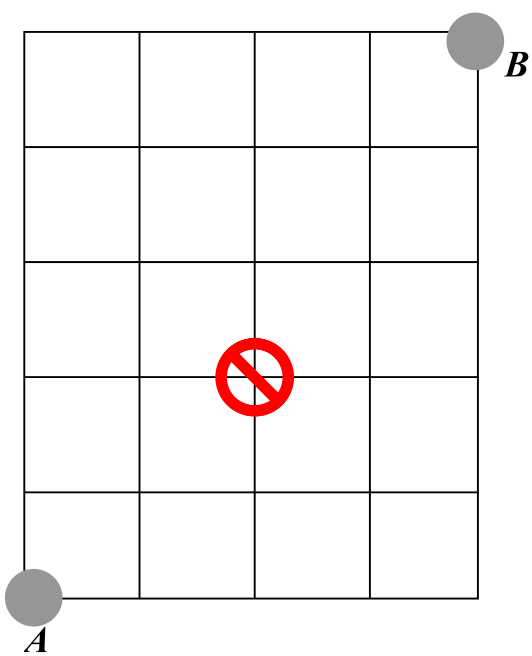
Example 6: Determine the number of ways you can reach *B* from *A* if the path taken must be in either north or east direction.

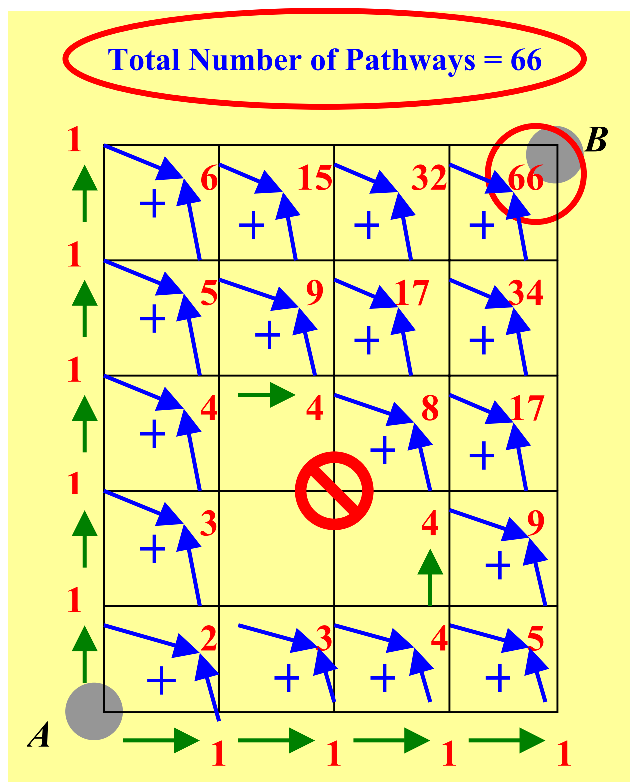
a.



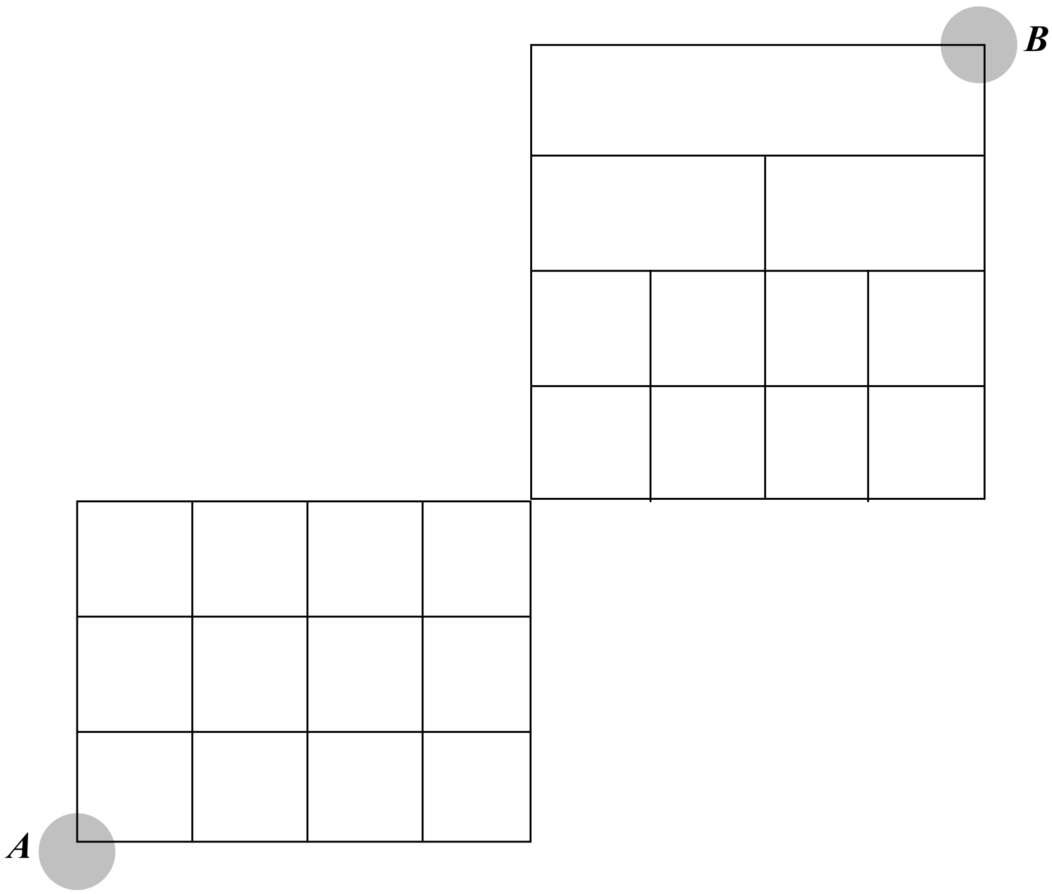


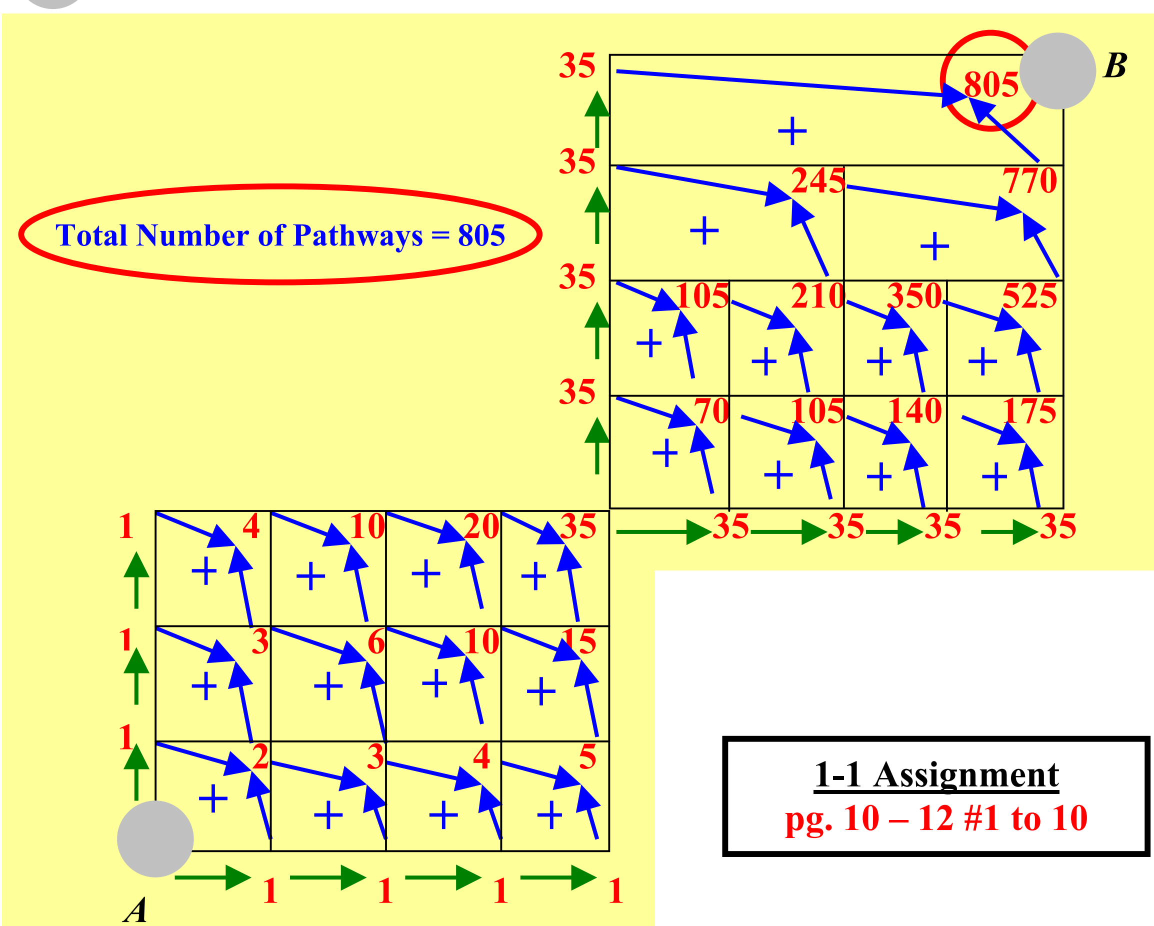
b.





c.





1-1 Assignment
pg. 10 – 12 #1 to 10

1-2: Generating and Using Sample Space

Sample Space: - a list of all possible outcomes of an experiment.

Complement: - probability of **ALL Non-Favourable** outcomes.

$P(A)$ = Probability of event A happening. $P(\overline{A})$ = Probability that event A will **NOT** happen.

In general, $$P(A) + P(\overline{A}) = 1$$ **or** $$P(\overline{A}) = 1 - P(A)$$

Example 1: Find the theoretical probability of rolling a 4 using a six-sided dice, and its compliment.

Sample Space of a dice = {1, 2, 3, 4, 5, 6}

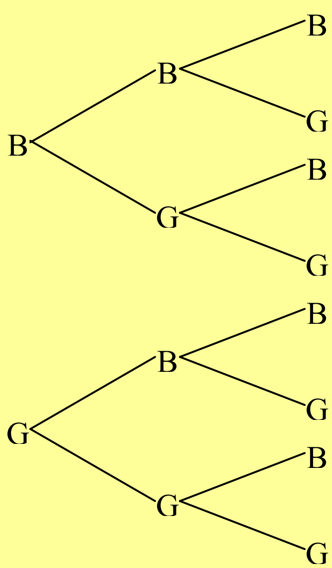
$\boldsymbol{P(4) = \frac{1}{6}}$ **(1 favourable outcome out of a total of 6 outcomes)**

$P(\overline{4}) = 1 - P(4) = 1 - \frac{1}{6}$ $\boldsymbol{P(\overline{4}) = \frac{5}{6}}$

Tree Diagram: - using branches to list all outcomes.
- following branches one path at a time to list all outcomes.
- its limitation lies in the fact that it can only handle individual events having **SMALL** number of outcomes.

Example 2: List the sample space of a family of 3 children. Let event A be at most having 2 girls, find $P(A)$ and $P(\overline{A})$.

1st child 2nd child 3rd child Outcomes



(B, B, B)
(B, B, G)
(B, G, B)
(B, G, G)
(G, B, B)
(G, B, G)
(G, G, B)
(G, G, G)

$P(A)$ = Probability of **at most** 2 girls
(no girls, 1 girl, and 2 girls)

$\boldsymbol{P(A) = \frac{7}{8}}$

(7 favourable outcomes out of a total of 8 outcomes)

OR we can say $P(A) = 1 - P$ (all girls)

$\boldsymbol{P(A) = 1 - \frac{1}{8}}$ $\boldsymbol{P(A) = \frac{7}{8}}$

(G, G, G) → $P(\overline{A}) = 1 - P(A) = 1 - \frac{7}{8}$ $\boldsymbol{P(\overline{A}) = \frac{1}{8}}$

Sample Space Table: useful when there are **TWO** items of **MANY** outcomes for each trial.

Example 3: List the sample space for the sums when 2 fair dice are thrown
a. Find the probability of rolling a sum of 8.
b. What is the probability of rolling at least an 8?

Second Dice / First Dice

Sum	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

a. $$P(8) = \frac{5}{36}$$

(5 favourable outcomes out of a total of 36 outcomes)

b. P (at least a sum of 8) = P (sum of 8 and higher)

$$P(\text{at least a sum of } 8) = \frac{15}{36}$$

$$P(\text{at least a sum of } 8) = \frac{5}{12}$$

Example 4: Without draw a tree diagram, determine the number outcomes will be in the sample space when 4 coins are tossed. What is the probability of tossing at least one head?

From Example 2, we noticed that when there were 3 events (3 children) of 2 outcomes (boy or girl) each, there were a total of 2 × 2 × 2 = 8 outcomes.

Therefore, it can be assumed that when there are 4 events (tossing four coins) of 2 outcomes (head or tail) each, there are a total of 2 × 2 × 2 × 2 = <u>16 outcomes</u>.

P (at least one head) = P (1 or 2 or 3 or 4 heads)
P (at least one head) = 1 – P (no head)
P (at least one head) = 1 – P (all tails)

$$P(\text{at least one head}) = 1 - \frac{1}{16}$$

(There is only 1 outcome out of a total of 16 outcomes to get all tails.)

$$P(\text{at least one head}) = \frac{15}{16}$$

<u>1-2 Assignment</u>: pg. 17 – 19 #1 to 10

1-3: The Fundamental Counting Principle

Counting Principle: - by multiplying the number of ways in each category of any particular outcome, we can find the total number if arrangements.

Example 1: How many outfits can you have if you have 3 different shirts, 2 pairs of pants, 4 pairs of socks, and 1 pair of shoes?

There are 4 categories: shirt, pants, socks and shoes.

3	×	2	×	4	×	1	=	**24 outfits**
shirts		pants		socks		shoes		

When the outcome involves many restrictions,
make sure to deal with the category that is the most restrictive first.

Example 2: How many 5-digits numbers are there if

a. there is no restriction?

1st digit		2nd digit		3rd digit		4th digit		5th digit
9	×	10	×	10	×	10	×	10
(1 to 9)		(0 to 9)		(0 to 9)		(0 to 9)		(0 to 9)

90000 numbers

b. they have to be divisible by 5?

1st digit		2nd digit		3rd digit		4th digit		5th digit
9	×	10	×	10	×	10	×	2
(1 to 9)		(0 to 9)		(0 to 9)		(0 to 9)		(0 or 5)
								restriction

18000 numbers

c. no digits are repeated?

1st digit		2nd digit		3rd digit		4th digit		5th digit
9	×	9	×	8	×	7	×	6
(1 to 9)		(0 to 9)		(0 to 9)		(0 to 9)		(0 to 9)
		− 1 digit		− 2 digits		− 3 digits		− 4 digits

27216 numbers

d. they have to be less than 40000 with non-repeated digits?

1st digit		2nd digit		3rd digit		4th digit		5th digit
3	×	9	×	8	×	7	×	6
(1 to 3)		(0 to 9)		(0 to 9)		(0 to 9)		(0 to 9)
		− 1 digit		− 2 digits		− 3 digits		− 4 digits

9072 numbers

e. at least two digits that are the same?

At least two digits the same = same 2 digits + same 3 digits + same 4 digits + same 5 digits

OR

At least two digits the same = Total number of ways with no restriction − No digits the same

= 90000 − 27216

62784 numbers

f. Find the probability of getting a 5-digits numbers if at least 2 digits have to be the same.

$$P\text{ (at least 2 same digits out of a 5-digits number)} = \frac{\text{Number of ways at least same 2 digits}}{\text{Total Number of ways}} = \frac{62784}{90000}$$

$$\mathbf{\frac{436}{625} \text{ or } 0.6976}$$

Example 3: In an Applied Math 30 Diploma Exam, there are 33 multiple-choice questions. Each question contains 4 choices. How many different combinations are there to complete the test, assuming all questions are answered? What is the probability that a student would guess all the answers correctly?

For each question, there are 4 ways (choices) to answer.

For 33 questions, there are $4 \times 4 \times 4 \times 4 \times 4 \times 4 \times \ldots$ } There are 33 factors of fours.

Total number of ways to answer the test = 4^{33} or $7.378697629 \times 10^{19}$ ways

P (guessing all the right answers) = $\frac{1}{4^{33}} = 1.35525272 \times 10^{-20}$

(There is a way better chance to win the lottery compared to guessing all the answers correctly!)

Example 4: Find the number of ways to select 2 individuals from a group of 6 people.

Let's suppose the names of the people are A, B, C, D, E and F.

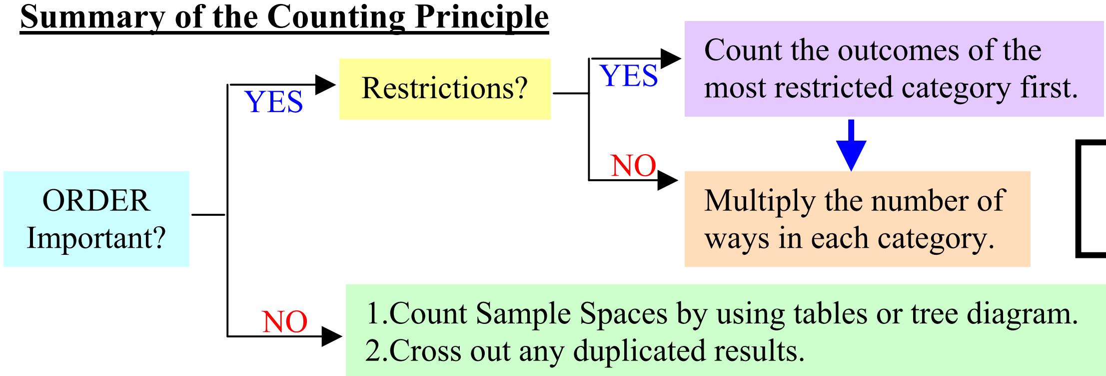
The sample space table would be

AB	BA	CA	DA	EA	FA
AC	**BC**	CB	DB	EB	FB
AD	**BD**	**CD**	DC	EC	FC
AE	**BE**	**CE**	**DE**	ED	FD
AF	**BF**	**CF**	**DF**	**EF**	FE

Order is NOT important! These repeated combinations do not need to be counted.

Total Number of ways = 15

Summary of the Counting Principle



1-3 Assignment: **pg. 23 – 25 #1 to 12**

1-4: Independent and Dependent Events

Independent Events: - when the outcome of one event does NOT affect the outcomes of the events follow.

$$P(\underbrace{A \text{ and } B}_{\text{Events } A \text{ and } B}) = P(A) \times P(B)$$

Example 1: What is the probability of rolling at most a "4" from a dice and selecting a diamond out of a standard deck of cards?

$$P(\text{at most "4" \& diamond}) = \frac{4}{6} \times \frac{13}{52}$$

Reduce before multiply

$$P(\text{at most "4" \& diamond}) = \frac{2}{3} \times \frac{1}{4} = \frac{2}{12}$$

$$\boldsymbol{P(\text{at most "4" \& diamond}) = \frac{1}{6}}$$

Example 2: Find the probability of getting at least one number correct out a 3-numbers combination lock with markings from 0 to 59 inclusive.

All numbers are Incorrect (No number is correct) is the Compliment Event of At Least ONE number is correct.

$$P(\text{All numbers are Incorrect}) = \frac{59}{60} \times \frac{59}{60} \times \frac{59}{60} = \left(\frac{59}{60}\right)^3$$ (There are 60 numbers from 0 to 59 inclusive.)

$$P(\text{At Least ONE number is correct}) = 1 - P(\text{All numbers are Incorrect})$$

$$P(\text{At Least ONE number is correct}) = 1 - \left(\frac{59}{60}\right)^3$$

$$\boldsymbol{P(\text{At Least ONE number is correct}) \approx 0.0492 \approx 4.92\%}$$

Example 3: What is the probability of drawing two aces if the first card is replaced (put back into the deck) before the second card is drawn?

$$P(\text{drawing 2 aces}) = \frac{4}{52} \times \frac{4}{52}$$

$$P(\text{drawing 2 aces}) = \frac{1}{13} \times \frac{1}{13}$$

$$\boldsymbol{P(\text{drawing 2 aces}) = \frac{1}{169} \approx 0.005917 \approx 0.592\%}$$

Dependent Events: - when the outcome of the first event AFFECTS the outcome(s) of subsequent event(s).

Example 4: What is the probability of drawing two hearts from a standard deck of 52 cards?

If the question doe NOT say, assume the card is NOT Replaced!

P (two hearts) = $\frac{13}{52} \times \frac{12}{51}$ ← One less heart remains in the deck after the 1st draw. ← One less card total remains in the deck after the 1st draw.

P (two hearts) = $\frac{1}{4} \times \frac{4}{17}$

$$P \text{ (two hearts)} = \frac{1}{17} \approx 0.0588 \approx 5.88\%$$

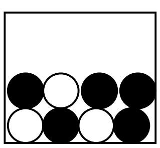
Example 5: About 40% of the population likes sci-fi movies, while 80% of the population likes comedies. If 35% of the population likes both sci-fi and comedies, determine whether preferences for sci-fi and comedies are independent events. Justify your answer mathematically.

P (sci-fi) = 40% = 0.40
P (comedies) = 80% = 0.80

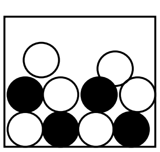
If sci-fi and comedies are independent events, then
P (sci-fi & comedies) = P (sci-fi) × P (comedies)
P (sci-fi & comedies) = 0.40 × 0.80
P (sci-fi & comedies) = **0.32 (independent events)**

Since the question indicates P (sci-fi & comedies) is 35% = 0.35, they should be considered as **DEPENDENT EVENTS**.

Example 6: There are two identical looking containers. Each contains different number of black and white balls of the same size. Container A has 8 balls in total and 3 of those are white. Container B has 10 balls in total and 4 of those are black. If a person has a choice to pick one ball out of these two containers, what is the probability of selecting a black ball?



Container A



Container B

***P* (black) = *P* (Container *A* & Black) OR *P* (Container *B* & Black)**

OR means ADD

P (black) = $P(A) \times P(\text{Black}) + P(B) \times P(\text{Black})$

P (black) = $\left(\frac{1}{2}\right)\times\left(\frac{5}{8}\right) + \left(\frac{1}{2}\right)\times\left(\frac{4}{10}\right)$

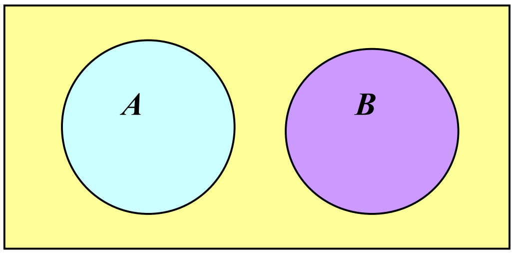
P (black) = $\frac{5}{16} + \frac{4}{20}$

$$P \text{ (black)} = \frac{41}{80} = 0.5125 = 51.25\%$$

1-4 Assignment: **pg. 31 – 33 #1 to 9, 11**

1-5: Mutually Exclusive Events

Mutually Exclusive Events: - when events A & B **CANNOT** occur at the **SAME TIME**.
- either event A occurs **OR** event B occurs.



Mutually Exclusive Events

$$P\,(A \textbf{ or } B) = P\,(A) + P\,(B)$$

NO Overlap

Example 1: A card is drawn from a standard deck of 52 cards. What is the probability that it is a red card or a spade?

Since there is no card that is BOTH a spade and red, the events are mutually exclusive.

$$P\,(\text{red}) = \frac{26}{52} = \frac{1}{2} \qquad P\,(\text{spade}) = \frac{13}{52} = \frac{1}{4}$$

$$P\,(\text{red or spade}) = P\,(\text{red}) + P\,(\text{spade})$$

$$P\,(\text{red or spade}) = \frac{1}{2} + \frac{1}{4}$$

$$\boldsymbol{P\,(\text{red or spade}) = \frac{3}{4}}$$

Example 2: Two dice are rolled. What is the probability that either the sum is 3 or the sum is 8?

Second Dice (columns); First Dice (rows)

Sum	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Since you can either roll a sum of 3 or a sum of 8, but NOT BOTH, the events are mutually exclusive.

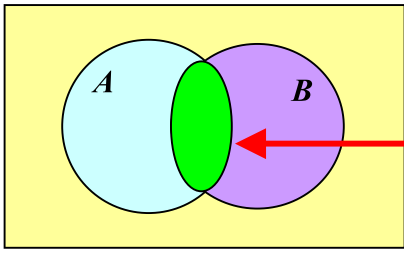
$$P\,(3) = \frac{2}{36} \qquad P\,(8) = \frac{5}{36}$$

$$P\,(3 \text{ or } 8) = P\,(3) + P\,(8)$$

$$P\,(3 \text{ or } 8) = \frac{2}{36} + \frac{5}{36}$$

$$\boldsymbol{P\,(3 \text{ or } 8) = \frac{7}{36}}$$

Non-Mutually Exclusive Events: - when events A & B **CAN** occur at the **SAME TIME**.
- the probability of the overlapping area, P (A and B), is subtracted.



Non-Mutually Exclusive Events

$$P(A \textbf{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Overlapping Area

$P(A \text{ and } B) = P(A) \times P(B)$ (if A and B are independent events)

Example 3: Two dice are rolled. What is the probability that one of the dice is 3 or the sum is 8?

		Second Dice					
	Sum	**1**	**2**	**3**	**4**	**5**	**6**
First Dice	**1**	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Since you can have sums of 8 that have 3 on one dice, these events are non-mutually exclusive.

$$P(\text{one of the dice is 3}) = \frac{10}{36}$$

(sum of 6 has two 3's – Not included)

$$P(8) = \frac{5}{36} \qquad P(\text{3 on one dice and sum is 8}) = \frac{2}{36}$$

$$P(\text{3 on one dice or sum of 8}) = P(3) + P(8) - P(\text{3 on one dice \& sum of 8})$$

$$P(\text{3 on one dice or sum of 8}) = \frac{10}{36} + \frac{5}{36} - \frac{2}{36}$$

$$P(\text{3 on one dice or sum of 8}) = \mathbf{\frac{13}{36}}$$

Example 4: Out of 200 people, 80 of them like sci-fi and 160 like comedies. 70 of them like both sci-fi and comedies. Find the probability that someone will like either sci-fi or comedies.

Since you can have people prefer BOTH sci-fi and comedies, these events are non-mutually exclusive.

$$P(\text{sci-fi}) = \frac{80}{200} \qquad P(\text{comedies}) = \frac{160}{200} \qquad P(\text{sci-fi and comedies}) = \frac{70}{200}$$

$$P(\text{sci-fi or comedies}) = P(\text{sci-fi}) + P(\text{comedies}) - P(\text{sci-fi and comedies})$$

$$P(\text{sci-fi or comedies}) = \frac{80}{200} + \frac{160}{200} - \frac{70}{200}$$

$$P(\text{sci-fi or comedies}) = \mathbf{\frac{170}{200} = \frac{17}{20} = 0.85 = 85\%}$$

Example 5: In our society, the probability of someone suffering from heart disease is 0.56 and the probability of developing cancer is 0.24. If these events are independent and non-mutually exclusive, what is the probability that you will suffer neither illness?

P (Heart Disease) = $P(H) = 0.56$ P (Cancer) = $P(C) = 0.24$

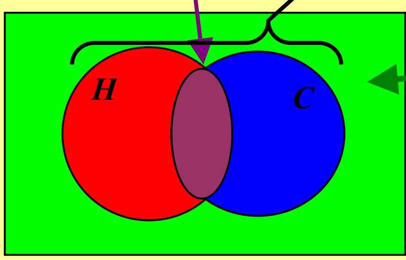
$P(H \text{ and } C) = P(H) \times P(C) = 0.56 \times 0.24$ (independent events)

$\boldsymbol{P(H \text{ and } C) = 0.1344}$

$P(H \text{ or } C) = P(H) + P(C) - P(H \text{ and } C)$ (non-mutually exclusive events)

$P(H \text{ or } C) = 0.56 + 0.24 - 0.1344$

$\boldsymbol{P(H \text{ or } C) = 0.6656}$



P (No Heart Disease NOR Cancer) = $P(\overline{H} \text{ and } \overline{C})$

Area OUTSIDE the circles

$P(\overline{H} \text{ and } \overline{C}) = 1 - P(H \text{ or } C)$

$P(\overline{H} \text{ and } \overline{C}) = 1 - 0.6656$

$\boldsymbol{P(\overline{H} \text{ and } \overline{C}) = 0.3344 = 33.44\%}$

Example 6: Two cards are drawn from a standard deck of 52 cards. What is the probability that both cards are face cards or both are hearts?

The question did not mention replacement. Therefore, we can assume there is NO replacement (Dependent Events).

Since you can have BOTH face cards and hearts, these events are non-mutually exclusive.

$P(\text{face cards}) = \frac{12}{52} \times \frac{11}{51} = \frac{11}{221}$ $P(\text{hearts}) = \frac{13}{52} \times \frac{12}{51} = \frac{1}{17}$

$P(\text{face cards and hearts}) = \frac{3}{52} \times \frac{2}{51} = \frac{1}{442}$ (K, Q, and J of Hearts)

$P(\text{face cards or hearts}) = P(\text{face cards}) + P(\text{hearts}) - P(\text{face cards and hearts})$

$P(\text{face cards or hearts}) = \frac{11}{221} + \frac{1}{17} - \frac{1}{442}$

$\boldsymbol{P(\text{face cards or hearts}) = \frac{47}{442} = 0.1063 = 10.63\%}$

1-5 Assignment: **pg. 37 – 39 #1 to 11**

