Unit 3: Matrices and Pathways

2-1: Matrix Operations

Matrix: - a rectangular array of numbers, known as elements, which are enclosed within brackets.

Name of the Matrix
(in capital letter)

$$B = \begin{array}{cl} \begin{bmatrix} 2 & 3 & 1 & 0 \\ -1 & 4 & 5 & 6 \\ 3 & -2 & -3 & -4 \end{bmatrix} & \begin{array}{l} \textbf{row 1} \\ \textbf{row 2} \\ \textbf{row 3} \end{array} \\ \begin{array}{cccc} \textbf{column 1} & \textbf{column 2} & \textbf{column 3} & \textbf{column 4} \end{array} & \end{array}$$

Dimension: - the size of the matrix (***number of rows by number of columns***).
- the example above is a 3×4 matrix. We can write B_{34}

Elements: - individual numbers on the matrix.
- represented by the name of the matrix (in lower case letter), followed by row number and then column number in subscripts
- the example above has 5 as an element in row 2 and column 3. We can write $b_{23} = 5$

Special Matrices

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 & 4 \\ 3 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

Row Matrix
(1 row only)

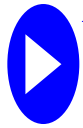
Column Matrix
(1 column only)

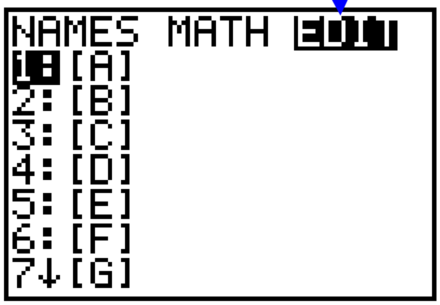
Square Matrix
(same number of rows as columns)

Using a Graphing Calculator to Operate with Matrices

A. To Enter a Matrix:

1. Press **2nd** **MATRX x^{-1}**

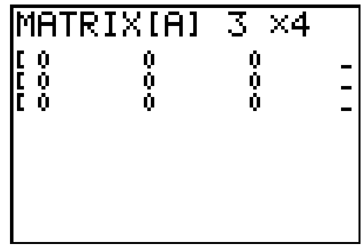
2. Use  to access EDIT



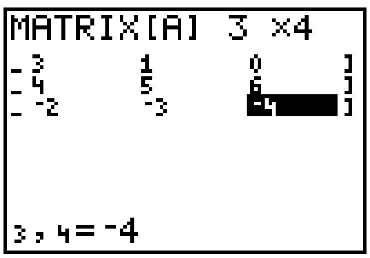
3. Select Option 1 if the desired name of the Matrix is [A]. Otherwise select other options for other names.

4. Press **ENTER**

5. Enter the dimensions of the matrix. (Using the first matrix on the previous page as example.)



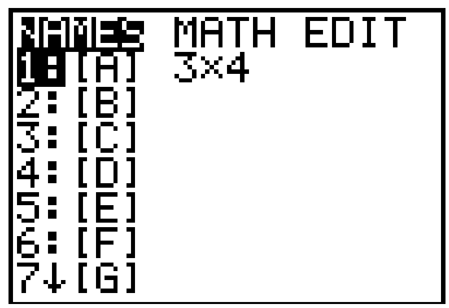
6. Enter the elements of the matrix (along each row).



7. Press **2nd** **QUIT MODE** when finished.

B. To Recall a Matrix from the Home Screen:

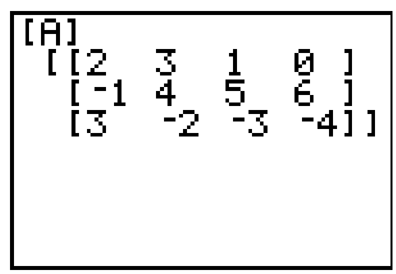
1. Press **2nd** **MATRX x^{-1}**



2. Select Option 1 if the desired matrix to be recalled is [A]. Otherwise select other options for other matrices.

3. Press **ENTER**

4. Press **ENTER** again to see the entire matrix on the home screen. (Highly recommended for matrices bigger than 3 × 3 to verify if there are any mistakes while entering elements.)



C. To Delete a Matrix:

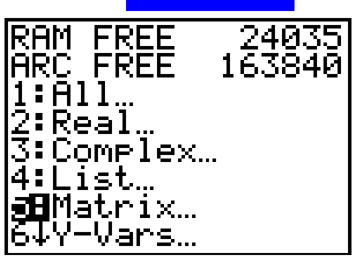
1. Press **2nd** **MEM +**

2. Select Option 2.

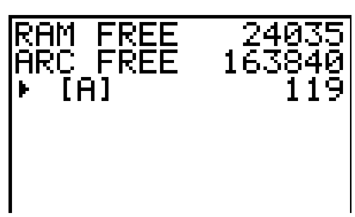
3. Press **ENTER**

4. Select Option 5.

5. Press **ENTER**



6. Press **INS DEL** next to the matrix that needs to be deleted.



Adding and Subtracting Matrices

- **can only be done if the matrices have the *SAME DIMENSIONS*.**
- **add or subtract each element with the CORRESPONDING element of another matrix.**

Example 1: For the matrices $A = \begin{bmatrix} 3 & 7 \\ -2 & 4 \\ 1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -2 \\ 0 & -3 \\ 4 & -5 \end{bmatrix}$, find

a. $A + B$

$$A + B = \begin{bmatrix} 3 & 7 \\ -2 & 4 \\ 1 & 6 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 0 & -3 \\ 4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 3+(-1) & 7+(-2) \\ (-2)+0 & 4+(-3) \\ 1+4 & 6+(-5) \end{bmatrix}$$

$$\boldsymbol{A + B = \begin{bmatrix} 2 & 5 \\ -2 & 1 \\ 5 & 1 \end{bmatrix}}$$

Verify with Calculator:

```
[A]+[B]
          [[2  5]
           [-2 1]
           [5  1]]
```

b. $A - B$

$$A - B = \begin{bmatrix} 3 & 7 \\ -2 & 4 \\ 1 & 6 \end{bmatrix} - \begin{bmatrix} -1 & -2 \\ 0 & -3 \\ 4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 3-(-1) & 7-(-2) \\ (-2)-0 & 4-(-3) \\ 1-4 & 6-(-5) \end{bmatrix}$$

$$\boldsymbol{A - B = \begin{bmatrix} 4 & 9 \\ -2 & 7 \\ -3 & 11 \end{bmatrix}}$$

Verify with Calculator:

```
[A]-[B]
          [[4  9 ]
           [-2 7 ]
           [-3 11]]
```

Multiplying Matrices with a Scalar (a single number)

- **multiply each element of the matrix with the scalar individually.**

Example 2: Convert the following chart to Canadian dollars. ($1 US = $1.64 Cdn)

Commodity Prices ($US)	Jan 1st, 2000	Jan 1st, 2001
Oil (per barrel)	$17.35	$28.92
Silver (per ounce)	$5.46	$6.21
Gold (per ounce)	$245.20	$231.48

Let C = Commodities Prices in $US D = Commodities Prices in $Cdn

$$C = \begin{array}{c c} & \begin{array}{cc} \text{Y2000} & \text{Y2001} \end{array} \\ \begin{array}{c} \text{O} \\ \text{S} \\ \text{G} \end{array} & \begin{bmatrix} 17.35 & 28.92 \\ 5.46 & 6.21 \\ 245.20 & 231.48 \end{bmatrix} \end{array} \qquad D = 1.64 \times C$$

(Always label the columns and row headings for matrices word problems.)

$$D = 1.64 \times \begin{array}{c c} & \begin{array}{cc} \text{Y2000} & \text{Y2001} \end{array} \\ \begin{array}{c} \text{O} \\ \text{S} \\ \text{G} \end{array} & \begin{bmatrix} 17.35 & 28.92 \\ 5.46 & 6.21 \\ 245.20 & 231.48 \end{bmatrix} \end{array} = \begin{array}{c c} & \begin{array}{cc} \text{Y2000} & \text{Y2001} \end{array} \\ \begin{array}{c} \text{O} \\ \text{S} \\ \text{G} \end{array} & \begin{bmatrix} 1.64 \times 17.35 & 1.64 \times 28.92 \\ 1.64 \times 5.46 & 1.64 \times 6.21 \\ 1.64 \times 245.20 & 1.64 \times 231.48 \end{bmatrix} \end{array}$$

$$D = \begin{array}{c c} & \begin{array}{cc} \textbf{Y2000} & \textbf{Y2001} \end{array} \\ \begin{array}{c} \textbf{O} \\ \textbf{S} \\ \textbf{G} \end{array} & \begin{bmatrix} \mathbf{28.45} & \mathbf{47.43} \\ \mathbf{8.95} & \mathbf{10.18} \\ \mathbf{402.13} & \mathbf{379.63} \end{bmatrix} \end{array}$$

Verify with Calculator:

```
1.64[C]
[[28.454   47.42…
 [8.9544   10.18…
 [402.128  379.6…
```

Use ▶ to see the rest of the matrix.

2-1 Assignments: pg. 55 – 59 #1 to 9

2-2: Multiplying Matrices

When multiplying matrices, we need to do "ROW on COLUMN"

- Multiply elements in a **row of the first matrix** with the elements in a **column of the second matrix**.
- **Add all the products.**
- **Repeat** steps above with another row and column.

If we let A be a 2 by 3 matrix and B be a 3 by 2 matrix, then $A_{23} \times B_{32}$ will be

Step 2

Step 1

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$$

Step 3

Step 4

Step 1 — Row 1 of A by Column 1 of B

Step 2 — Row 1 of A by Column 2 of B

Step 3 — Row 2 of A by Column 1 of B

Step 4 — Row 2 of A by Column 2 of B

new dimensions

Notice the product of $A_{2\,3} \times B_{3\,2}$ is a 2 by 2 matrix (P_{22})

match

The two inner dimensions (column number of the first matrix and row number of the second matrix) must be the SAME.

Matrices Multiplications "Row by Column"

New Dimensions of the Product

$$A_{m\,n} \times B_{n\,p} = AB_{mp}$$

must be the SAME!

Product's New Dimensions

In Matrices

$$A \times B \neq B \times A$$

Example 1: Given $A = \begin{bmatrix} 5 & 4 & 0 & 1 \\ -2 & 3 & -1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ 4 & 0 \\ 4 & 3 \\ 8 & 1 \end{bmatrix}$, find

a. $A \times B$ (show you work)

$$A_{24} \times B_{42} = Product_{22}$$

$$\begin{bmatrix} 5 & 4 & 0 & 1 \\ -2 & 3 & -1 & 6 \end{bmatrix} \times \begin{bmatrix} -3 & 2 \\ 4 & 0 \\ 4 & 3 \\ 8 & 1 \end{bmatrix} = \begin{bmatrix} (5)(-3)+(4)(4)+(0)(4)+(1)(8) & (5)(2)+(4)(0)+(0)(3)+(1)(1) \\ (-2)(-3)+(3)(4)+(-1)(4)+(6)(8) & (-2)(2)+(3)(0)+(-1)(3)+(6)(1) \end{bmatrix}$$

$$\mathbf{A_{24} \times B_{42} = \begin{bmatrix} 9 & 11 \\ 62 & -1 \end{bmatrix}}$$

b. $B \times A$ (use calculator)

1. Enter Matrix A

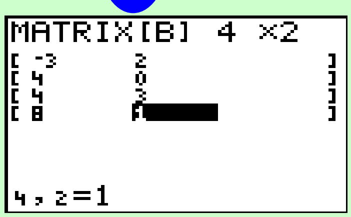
a. Press **2nd** **MATRX x^{-1}**

b. Use ▶ to access EDIT

2. Enter Matrix B

a. Press **2nd** **MATRX x^{-1}**

b. Use ▶ to access EDIT

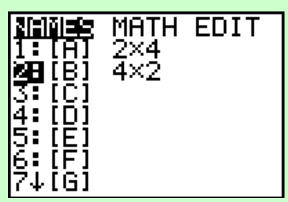


c. When finished., press **2nd** **QUIT MODE**

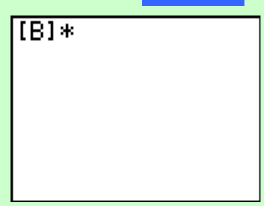
3. Recall Matrix B to multiply Matrix A

a. Press **2nd** **MATRX x^{-1}**

b. Select Option 2

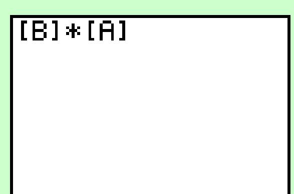


c. Press **×**

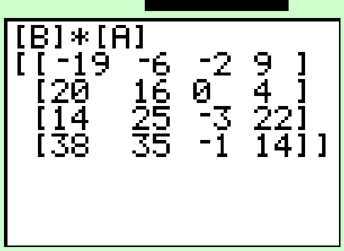


d. Press **2nd** **MATRX x^{-1}**

e. Select Option 1



f. Press **ENTER**



$$B_{42} \times A_{24} = Product_{44}$$

$$\begin{bmatrix} -19 & -6 & -2 & 9 \\ 20 & 16 & 0 & 4 \\ 14 & 25 & -3 & 22 \\ 38 & 35 & -1 & 14 \end{bmatrix}$$

Example 2: Competing companies, *Barq* and *A&W*, sells root beer in 355 mL cans, 750 mL cans, and 2L bottle at prices of \$1.00, \$1.50, and \$2.75 respectively. The sales table for the two companies for the month of May at a Safeway supermarket is as follows.

Root Beer Sales	355 mL	750 mL	2 L
Barq	850	500	400
A&W	900	450	525

What is the total revenue for each company?

First, we should create matrix *C* for the sales table and matrix D for the unit prices.

$$\text{Matrix } C = \begin{array}{r} \\ \text{Barq} \\ \text{A \& W} \end{array}\begin{array}{c} \begin{array}{ccc} 355\text{mL} & 750\text{mL} & 2\text{L} \end{array} \\ \begin{bmatrix} 850 & 500 & 400 \\ 900 & 450 & 525 \end{bmatrix} \end{array} \qquad \text{Matrix } D = \begin{array}{r} 355\text{mL} \\ 750\text{mL} \\ 2\text{L} \end{array}\begin{array}{c} \text{Unit Prices} \\ \begin{bmatrix} \$1.00 \\ \$1.50 \\ \$2.75 \end{bmatrix} \end{array}$$

If we want to keep Matrix *C* as a 2 × 3 table, Matrix *D* **(Unit Prices) must be a column matrix** of 3 × 1. Otherwise, we cannot multiply them. **Besides, the items in the column heading of the first matrix must match the item in the row heading of the second matrix.**

$C_{23} \times D_{31} = Revenue_{21}$

$$Revenue_{21} = \begin{array}{r} \text{Barq} \\ \text{A \& W} \end{array}\begin{array}{c} \begin{array}{ccc} 355\text{mL} & 750\text{mL} & 2\text{L} \end{array} \\ \begin{bmatrix} 850 & 500 & 400 \\ 900 & 450 & 525 \end{bmatrix} \end{array} \times \begin{array}{r} 355\text{mL} \\ 750\text{mL} \\ 2\text{L} \end{array}\begin{array}{c} \text{Unit Prices} \\ \begin{bmatrix} \$1.00 \\ \$1.50 \\ \$2.75 \end{bmatrix} \end{array}$$

$$= \begin{array}{r} \text{Barq} \\ \text{A \& W} \end{array}\begin{array}{c} \text{Revenue} \\ \begin{bmatrix} 850(\$1.00) + 500(\$1.50) + 400(\$2.75) \\ 900(\$1.00) + 450(\$1.50) + 525(\$2.75) \end{bmatrix} \end{array}$$

$$\boldsymbol{Revenue_{21}} = \begin{array}{r} \textbf{Barq} \\ \textbf{A \& W} \end{array}\begin{array}{c} \textbf{Revenue} \\ \begin{bmatrix} \$2700.00 \\ \$3018.75 \end{bmatrix} \end{array}$$

Verify with Calculator:

```
[C]*[D]
     [[2700    ]
      [3018.75]]
```

Example 3: The following table shows the prices of several clothing item in various store.

Men's Clothes	*GAP*	*Le Chateau*	*Club Monaco*	*Value Village*
Shirt	\$40.00	\$35.00	\$52.00	\$6.00
Pants	\$65.00	\$50.00	\$60.00	\$10.00
Underwear	\$10.00	\$8.00	\$12.00	\$1.50

If *GAP* and *Club Monaco* have a sale for 20% and 25% off respectively, **using matrices**, create a table of these 4 stores that show their final prices.

We can first create matrix E (3 × 4) for the price table. But to multiply E with another matrix and the end product (new prices) stays as (3 × 4), we **must create a matrix F that is a 4 × 4**. Since *GAP* and *Club Monaco* has a 20% and 25% off, and the other stores has no sales, **matrix F must have 0, 1, 0.8, and 0.75 as its elements.**

$$E_{34} \times F_{44} = \textit{Final Prices}_{34}$$

$$E_{34} = \begin{array}{c} \\ S \\ P \\ U \end{array}\begin{array}{c} \begin{array}{cccc} G & LC & CM & VV \end{array} \\ \begin{bmatrix} 40 & 35 & 52 & 6 \\ 65 & 50 & 60 & 10 \\ 10 & 8 & 12 & 1.5 \end{bmatrix} \end{array}$$

$$F_{44} = \text{From Old Price} \begin{array}{c} \\ G \\ LC \\ CM \\ VV \end{array}\begin{array}{c} \text{To New Price} \\ \begin{array}{cccc} G & LC & CM & VV \end{array} \\ \begin{bmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.75 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$E_{34} \times F_{44} = \begin{array}{c} \\ S \\ P \\ U \end{array}\begin{array}{c} \begin{array}{cccc} G & LC & CM & VV \end{array} \\ \begin{bmatrix} 40 & 35 & 52 & 6 \\ 65 & 50 & 60 & 10 \\ 10 & 8 & 12 & 1.5 \end{bmatrix} \end{array} \times \text{From Old Price} \begin{array}{c} \\ G \\ LC \\ CM \\ VV \end{array}\begin{array}{c} \text{New Prices} \quad \begin{array}{cccc} G & LC & CM & VV \end{array} \\ \begin{bmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.75 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The zeros represent the price changes between stores are impossible

Row 1 by Column 1
40(0.8) +35(0) + 52(0) + 6 (0)

Row 2 by Column 3
65(0) +50(0) + 60(0.75) + 10(0)

$$= \begin{array}{c} \\ S \\ P \\ U \end{array}\begin{array}{c} \begin{array}{cccc} G & LC & CM & VV \end{array} \\ \begin{bmatrix} 40(0.8) & 35(1) & 52(0.75) & 6(1) \\ 65(0.8) & 50(1) & 60(0.75) & 10(1) \\ 10(0.8) & 8(1) & 12(0.75) & 1.50(1) \end{bmatrix} \end{array}$$

Row 3 by Column 2
10(0) + 8(1) + 12(0) + 1.5(0)

$$\textit{Final Price} = \begin{array}{c} \\ S \\ P \\ U \end{array}\begin{array}{c} \begin{array}{cccc} G & LC & CM & VV \end{array} \\ \begin{bmatrix} \$32 & \$35 & \$39 & \$6 \\ \$52 & \$50 & \$45 & \$10 \\ \$8 & \$8 & \$9 & \$1.50 \end{bmatrix} \end{array}$$

Verify with Calculator:

```
[E]*[F]
[[32 35 39 6  ]
 [52 50 45 10 ]
 [8  8  9  1.5]]
```

2-2 Assignment
pg. 64 – 67 #1 to 10

2-3: Solving Network Problems with Matrices

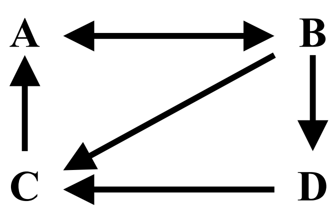
Network: - a diagram to represent the number of paths (kinds of relationships) between endpoints.

Network Matrix: - a matrix that represent the number of ***DIRECT*** paths between endpoints
- the rows and columns of the matrix represent "from endpoints → to endpoints".
- **0** is used to indicate **NO direct contact** while **1** is used to indicate **direct contact**.
- **NETWORK matrix is always a SQUARE Matrix.**

$$\textbf{Network Matrix } (N) = \quad \textbf{From} \quad \begin{array}{c} \\ \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \textbf{To} \\ A \quad B \quad C \quad D \\ \left[\begin{array}{c} \\ \text{Number of } \textit{Direct} \\ \text{Paths "from and to"} \\ \text{all Endpoints} \\ \\ \end{array} \right] \end{array}$$

N^n = **Network Matrix between endpoints for n paths or $(n-1)$ intermediates**

Example 1: The following diagram shows the network between 4 people. Single-headed arrow indicates one-way contact and double-headed arrow indicates two-way contact.



a. Create a table to show the number of direct paths one person can contact the other. Represent the table by Matrix M.

		To			
		A	**B**	**C**	**D**
From	**A**	0	1	0	0
	B	1	0	1	1
	C	1	0	0	0
	D	0	0	1	0

$$M = \text{From} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \text{To} \\ A \quad B \quad C \quad D \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

b. Create a table that shows the number of ways between these four people if there was **exactly** one intermediate person (2 paths) using matrix *M*.

For 1 intermediate = 2 paths, ***n* = 2**

Using Calculator

```
[A]²
      [[1 0 1 1]
       [1 1 1 0]
       [0 1 0 0]
       [1 0 0 0]]
```

$$M^2 = \text{From}\ \begin{array}{c|cccc|} & \multicolumn{4}{c}{\textbf{To}} \\ & A & B & C & D \\ A & 1 & 0 & 1 & 1 \\ B & 1 & 1 & 1 & 0 \\ C & 0 & 1 & 0 & 0 \\ D & 1 & 0 & 0 & 0 \end{array}$$

c. Create a table that shows the number of ways between these four people if there were **exactly** two intermediate people (3 paths) using matrix *M*.

For 2 intermediates = 3 paths, ***n* = 3**

Using Calculator

```
[A]^3
      [[1 1 1 0]
       [2 1 1 1]
       [1 0 1 1]
       [0 1 0 0]]
```

$$M^3 = \text{From}\ \begin{array}{c|cccc|} & \multicolumn{4}{c}{\textbf{To}} \\ & A & B & C & D \\ A & 1 & 1 & 1 & 0 \\ B & 2 & 1 & 1 & 1 \\ C & 1 & 0 & 1 & 1 \\ D & 0 & 1 & 0 & 0 \end{array}$$

d. Create a table that shows the number of ways between these four people if there were **at most** two intermediate people by using matrix *M*.

At most 2 intermediates = at most 3 paths,

***n* = 1 or 2 or 3**

Using Calculator

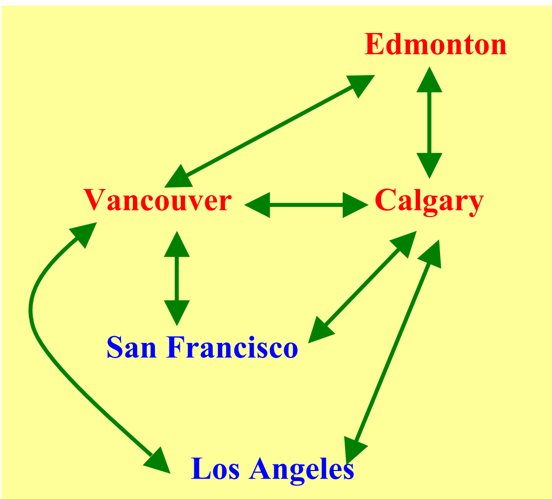
```
[A]+[A]²+[A]^3
      [[2 2 2 1]
       [4 2 3 2]
       [2 1 1 1]
       [1 1 1 0]]
```

$$M + M^2 + M^3 = \text{From}\ \begin{array}{c|cccc|} & \multicolumn{4}{c}{\textbf{To}} \\ & A & B & C & D \\ A & 2 & 2 & 2 & 1 \\ B & 4 & 2 & 3 & 2 \\ C & 2 & 1 & 1 & 1 \\ D & 1 & 1 & 1 & 0 \end{array}$$

Example 2: The western US and Canadian routes for Air Canada are as follows.

- Calgary to Vancouver (Direct Flight)
- Calgary to San Francisco (Direct Flight)
- Calgary to Los Angeles (Direct Flight)
- Vancouver to San Francisco (Direct Flight)
- Vancouver to Los Angeles (Direct Flight)
- Edmonton to Vancouver (Direct Flight)
- Edmonton to San Francisco (Stopover at Calgary/Vancouver)
- Edmonton to Los Angeles (Stopover at Calgary/Vancouver)

a. Draw a network diagram



b. Create a network matrix of direct flight.

$$B = \text{From}\ \begin{array}{c|c} & \text{To} \\ & \begin{array}{ccccc} \text{E} & \text{C} & \text{V} & \text{SF} & \text{LA} \end{array} \\ \begin{array}{c} \text{E} \\ \text{C} \\ \text{V} \\ \text{SF} \\ \text{LA} \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

c. Find the number of ways a person can fly from Edmonton to San Francisco with **exactly** one stopover (2 paths).

For 1 stopover = 2 paths, $n = 2$

$$B^2 = \text{From}\ \begin{array}{c|c} & \text{To} \\ & \begin{array}{ccccc} \text{E} & \text{C} & \text{V} & \text{SF} & \text{LA} \end{array} \\ \begin{array}{c} \text{E} \\ \text{C} \\ \text{V} \\ \text{SF} \\ \text{LA} \end{array} & \begin{bmatrix} 2 & 1 & 1 & 2 & 2 \\ 1 & 4 & 3 & 1 & 1 \\ 1 & 3 & 4 & 1 & 1 \\ 2 & 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 & 2 \end{bmatrix} \end{array}$$

There are <u>two ways</u> to fly from Edmonton to San Francisco with exactly one stopover.
(E $\rightarrow$ C $\rightarrow$ SF and E $\rightarrow$ V $\rightarrow$ SF)

d. Find the number of ways a person can fly from Edmonton to Los Angeles with **at most** two stopovers (3 paths).

For at most 2 stopovers = at most 3 paths, n = 1 or 2 or 3

$$B + B^2 + B^3 = \text{From}\ \begin{array}{c|c} & \text{To} \\ & \begin{array}{ccccc} \text{E} & \text{C} & \text{V} & \text{SF} & \text{LA} \end{array} \\ \begin{array}{c} \text{E} \\ \text{C} \\ \text{V} \\ \text{SF} \\ \text{LA} \end{array} & \begin{bmatrix} 4 & 9 & 9 & 4 & 4 \\ 9 & 10 & 11 & 9 & 9 \\ 9 & 11 & 10 & 9 & 9 \\ 4 & 9 & 9 & 4 & 4 \\ 4 & 9 & 9 & 4 & 4 \end{bmatrix} \end{array}$$

There are <u>four ways</u> to fly from Edmonton to Los Angeles with at most two stopovers.
(E $\rightarrow$ C $\rightarrow$ LA and E $\rightarrow$ V $\rightarrow$ LA)
(E $\rightarrow$ C $\rightarrow$ V $\rightarrow$ LA and E $\rightarrow$ V $\rightarrow$ C $\rightarrow$ LA)

<u>2-3 Assignment</u>: pg. 76 – 79 #1 to 7

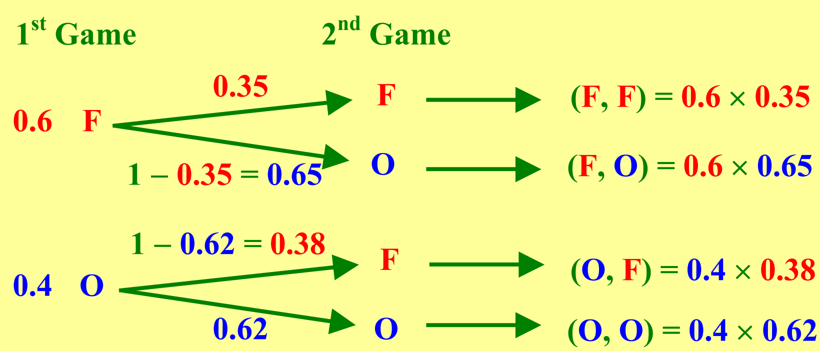
2-4: Solving Transition Problems with Matrix

Transition Matrix (T): a square matrix that shows how the probability of one event is **dependent** on another events occurring.

$$P_n = P_o \times T^n$$
$$P_o = \textbf{Initial Probability Matrix}$$
$$T = \textbf{Transition Matrix}$$
$$n = \textbf{Number of Transition after the first transition}$$
$$P_n = \textbf{Probability Matrix after the first transition}$$

Example 1: The Flames and Oilers are in the playoff. Their historical statistics for the first playoff has the winning probabilities of 0.6 and 0.4 against each other respectively. The probabilities that the Flames and Oilers taking the next playoff game after they win the first game of the series are 0.35 and 0.62 respectively.

a. Draw a tree diagram to determine the probabilities of each team winning the second playoff game.



***P* (Flames take 2nd Game)** = (F, F) + (O, F) = (0.6 × 0.35) + (0.4 × 0.38) = **0.362**
***P* (Oilers take 2nd Game)** = (F, O) + (O, O) = (0.6 × 0.65) + (0.4 × 0.62) = **0.638**

b. Write the initial probability matrix with proper labels.

Initial Probability Matrix $= P_o$

$$\begin{array}{cc} & \begin{array}{cc} \textbf{F} & \textbf{O} \end{array} \\ P_o = \textbf{Win} & \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \end{array}$$

c. Write the transition matrix with proper labels.

Transition Matrix = FROM First playoff game TO Second playoff game

$$T = \text{First playoff game} \quad \begin{array}{c} \\ \\ \mathbf{F} \\ \mathbf{O} \end{array} \begin{array}{c} \textbf{Next playoff game} \\ \begin{array}{cc} \mathbf{F} & \mathbf{O} \end{array} \\ \begin{bmatrix} \mathbf{0.35} & \mathbf{0.65} \\ \mathbf{0.38} & \mathbf{0.62} \end{bmatrix} \end{array}$$

$1 - 0.35$

$1 - 0.62$

(0.38 represents the probability that Oilers won the first game of the series and Flames won the second game of the series.)

d. Find the probability matrix for the teams to win second playoff game in the series.

***n* = 1 (after the first playoff game)** $\quad P_1 = P_0 \times T$

$$P_1 = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \times \begin{bmatrix} 0.35 & 0.65 \\ 0.38 & 0.62 \end{bmatrix}$$

$$\begin{array}{ccc} & \mathbf{F} & \mathbf{O} \\ P_1 = & [\mathbf{0.362} & \mathbf{0.638}] \end{array}$$

The probability of Flames and Oilers of winning the second playoff game regardless of the outcome in first game is 0.362 and 0.638 respectively.

(Notice this is the same as the answer in part a. The calculations done on the tree diagram are the same as the row by column of the matrix multiplication.)

e. Find the probability matrix for the teams to win their seventh playoff game in the series.

***n* = 6 (after the sixth playoff game)** $\quad P_6 = P_0 \times T^6$

$$P_6 = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \times \begin{bmatrix} 0.35 & 0.65 \\ 0.38 & 0.62 \end{bmatrix}^6$$

$$\begin{array}{ccc} & \mathbf{F} & \mathbf{O} \\ P_6 = & [\mathbf{0.3689} & \mathbf{0.6311}] \end{array}$$

The probability of Flames and Oilers of winning the 7th playoff game regardless of the outcomes in any previous games is 0.3689 and 0.6311 respectively.

Example 2: In the year 2000, out of all the working Canadian, 3% work in the USA, 2% work in other foreign countries, and the rest stays in Canada. Out of the Canadians working in the States, 75% says they will work in the US next year, while 20% will go back to Canada and work. The rest will have jobs in other foreign countries. Out of the Canadians working in foreign countries, 85% says they will work in the foreign countries next year, while 11% will go to work in the US, and the remaining will return to Canada to work. Among the Canadians who work at home, 10% says they will move to the US to work next year, while 86% says they will remain in Canada, and the rest indicate that they will move to a foreign country to work.

a. Write out the initial probability and the transitional matrices

First we have to organize the information above onto tables.

Probabilities of the Location of Working Canadians in the Year 2000

Canada	**USA**	**Other Countries**
1 – 0.03 –0.02 = **0.95**	**0.03**	**0.02**

Probabilities of the Migration of the Canadian Workforce from Year to Year

		To		
		Canada	**USA**	**Other Countries**
From	**Canada**	**0.86**	**0.10**	1 – 0.86 – 0.10 = **0.04**
	USA	**0.20**	**0.75**	1 – 0.20 – 0.75 = **0.05**
	Other Countries	1 – 0.11 – 0.85 = **0.04**	**0.11**	**0.85**

$$\begin{array}{cccc} & \textbf{CAN} & \textbf{USA} & \textbf{Others} \\ P_0 = & [\mathbf{0.95} & \mathbf{0.03} & \mathbf{0.02}] \end{array}$$

$$T = \begin{array}{cc} & \textbf{To} \\ & \begin{array}{ccc} \textbf{CAN} & \textbf{USA} & \textbf{Others} \end{array} \\ \textbf{From} \begin{array}{r} \textbf{CAN} \\ \textbf{USA} \\ \textbf{Others} \end{array} & \begin{bmatrix} \mathbf{0.86} & \mathbf{0.10} & \mathbf{0.04} \\ \mathbf{0.20} & \mathbf{0.75} & \mathbf{0.05} \\ \mathbf{0.04} & \mathbf{0.11} & \mathbf{0.85} \end{bmatrix} \end{array}$$

b. Find out the probabilities of where Canadians will work in 2001.

P_o = Year 2000 P_1 = Year 2001 $P_1 = P_o \times T$

$$P_1 = \begin{array}{c} \text{CAN USA Others} \\ \begin{bmatrix} 0.95 & 0.03 & 0.02 \end{bmatrix} \end{array} \times \text{From} \begin{array}{c} \\ \text{CAN} \\ \text{USA} \\ \text{Others} \end{array} \begin{array}{c} \text{To} \\ \text{CAN USA Others} \\ \begin{bmatrix} 0.86 & 0.10 & 0.04 \\ 0.20 & 0.75 & 0.05 \\ 0.04 & 0.11 & 0.85 \end{bmatrix} \end{array}$$

CAN USA Others

$$\mathbf{P_1 = [0.8238 \quad 0.197 \quad 0.0565]}$$

c. Determine the location demographics of the Canadian work force in the year 2010.

P_o = Year 2000 P_{10} = Year 2010 $P_{10} = P_o \times T^{10}$

$$P_{10} = \begin{array}{c} \text{CAN USA Others} \\ \begin{bmatrix} 0.95 & 0.03 & 0.02 \end{bmatrix} \end{array} \times \text{From} \begin{array}{c} \\ \text{CAN} \\ \text{USA} \\ \text{Others} \end{array} \begin{array}{c} \text{To} \\ \text{CAN USA Others} \\ \begin{bmatrix} 0.86 & 0.10 & 0.04 \\ 0.20 & 0.75 & 0.05 \\ 0.04 & 0.11 & 0.85 \end{bmatrix}^{10} \end{array}$$

CAN USA Others

$$\mathbf{P_{10} = [0.5138 \quad 0.2872 \quad 0.1990]}$$

d. A steady state is defined when the probabilities matrix, P_n, is the same as its previous probabilities matrix, P_{n-1}. When will the location demographics of the Canadian workforce stabilize? Find the percentage of the work force Canada will be losing to other countries including the USA?

By trial and error, *we* find that $P_{33} \approx P_{32}$ (to the ten-thousandth of a decimal place)

```
[[.4823606822 ...
[C]*[D]^31
[[.4822785981 ...
[C]*[D]^32
[[.4822120902 ...
[C]*[D]^33
[[.4821581976 ...
```

CAN USA Others

$$\mathbf{P_{33} = [0.4822 \quad 0.2922 \quad 0.2256]}$$

CAN USA Others

$$\mathbf{P_{32} = [0.4822 \quad 0.2922 \quad 0.2256]}$$

The location demographics of the Canadian workforce will stabilize in 2033. At that time, we will lose (1 – 0.4822) = 51.78% of our workforce to other countries including the USA.

<u>2-4 Assignment</u>: pg. 83 – 85 #1 to 10