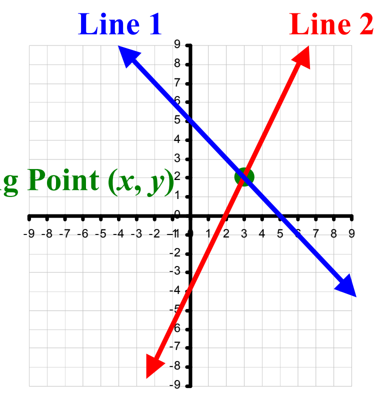
Chapter 10: Systems of Equations and Inequalities

10-1 Systems of Equations

System of Linear Equations: - two or more linear equations on the same coordinate grid.

Solution of a System of Linear Equations:

- the **intersecting point** of two or more linear equations
- on the Cartesian Coordinate Grid, the solution contains two parts: the ***x*-coordinate** and the ***y*-coordinate** (can be expressed as an **Ordered Pair**)



Solution of the System of Linear Equations

We can find the solution of a system of linear equation graphing manually or using a graphing calculator.

Example 1: Find the solution of the system of equations, $\begin{cases} x + 2y = 6 \\ x - y = 3 \end{cases}$ by graphing manually.

For Line 1:

$x + 2y = 6$

$2y = -x + 6$

$y = \dfrac{-x+6}{2}$

$y = -\tfrac{1}{2}x + 3$

***y*-int = (0, 3)**

slope $= \dfrac{-1}{2} = \dfrac{\text{1 Down}}{\text{2 Right}}$

For Line 2:

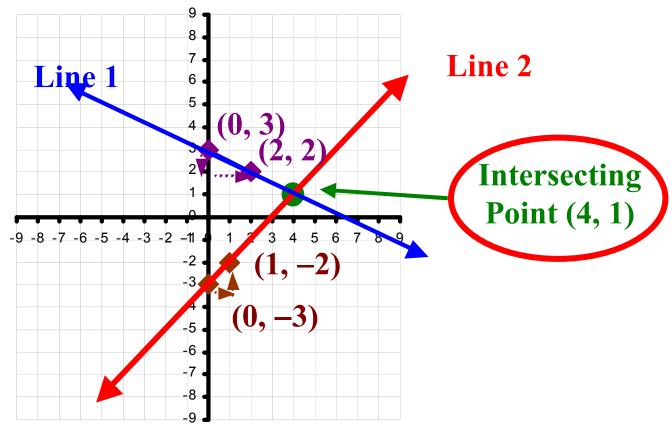
$x - y = 3$

$-y = -x + 3$

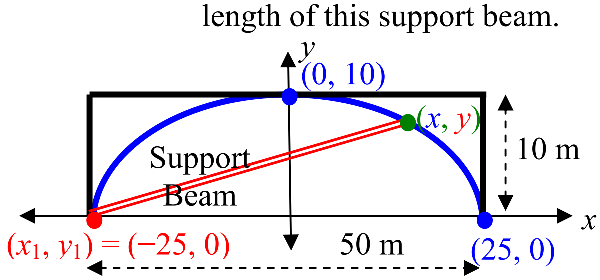
$y = x - 3$

***y*-int = (0, –3)**

slope $= \dfrac{1}{1} = \dfrac{\text{1 Up}}{\text{1 Right}}$



Example 2: The underside of a bridge forms a parabola with an equation $y = -\frac{2}{125}x^2 + 10$, where the origin is located on the ground directly below the midpoint of the span. A metal beam with a slope of $\frac{1}{5}$ from the bottom left of the bridge is used for extra support during its repair. Determine the length of this support beam.



Find equation of the beam:

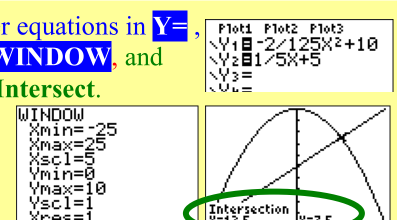
Given: $m = \frac{1}{5}$ and $(-25, 0)$

$y = mx + b \rightarrow (0) = \frac{1}{5}(-25) + b$

$b = 5$

Eq. of the beam: $y = \frac{1}{5}x + 5$

Enter equations in **Y=**, set **WINDOW**, and run **Intersect**.



$(x_2, y_2) = (12.5, 7.5)$

First, we have to label the coordinates on the parabola and the support beam. By finding the intersecting point (x, y), we can then use the distance formula to find the length of the beam.

Beam Length $= d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d = \sqrt{(12.5 - (-25))^2 + (7.5 - 0)^2}$

$d = \sqrt{(37.5)^2 + (7.5)^2}$

$d = 38.243$ m

Solving Systems of Linear Equations by Substitution:

When using the **substitution method** to solve a system of linear equations:

1. **Isolate a variable from one equation. (Always pick the variable with 1 as a coefficient.)**
2. **Substitute the resulting expression into that variable of the other equation.**
3. **Solve for the other variable.**
4. **Substitute the result from the last step into one of the original equation and solve for the remaining variable.**

Example 3: Using the substitution method, solve the systems of equations, $\begin{cases} 5x + y = -17 \\ 3y - 4x = 6 \end{cases}$ algebraically. Verify the solutions with the graphing calculator.

Isolate y from the first equation (a variable with 1 as a coefficient).

$$5x + y = -17$$
$$y = -5x - 17$$

Substitute expression into y in the second equation.

$$3y - 4x = 6$$
$$3(-5x - 17) - 4x = 6$$
$$-15x - 51 - 4x = 6$$
$$-19x = 6 + 51$$
$$-19x = 57$$
$$x = \frac{57}{-19}$$

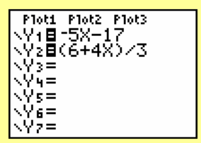
$x = -3$

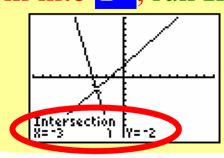
Solve for the remaining variable. Pick the easier equation of the two.

$$5x + y = -17$$
$$5(-3) + y = -17$$
$$-15 + y = -17$$
$$y = -17 + 15$$

$y = -2$

Verify with graphing calculator. Rearrange equation first. Enter them into **Y=**, run **Intersect**





Solving Systems of Linear Equations by Elimination:

Since the substitution method is only useful when an equation has 1 or −1 as the numerical coefficient, we need another way to solve other systems of linear equations.

Elimination by Addition: - most useful when both equations has the same like terms with opposite signs.

Elimination by Subtraction: - most useful when both equations has exactly the same like terms.

Elimination by Multiplication: - most useful when neither equations has the same like terms.
- by multiplying different numbers (factors of their LCM) on each equation, we can change these equations into their equivalent form with the same like terms.

Example 4: Solve the system of linear equations $\begin{cases} 3x + 4y = 18 \\ 2x - 3y = -5 \end{cases}$ by elimination.

Method 1: Eliminating x

LCM of $3x$ and $2x$ = $6x$

(Multiply Equation 1 by **2** to obtain $6x$)
(Multiply Equation 2 by **3** to obtain $6x$)

$2 \times (3x + 4y = 18) \longrightarrow 6x + 8y = 36$
$3 \times (2x - 3y = -5) \longrightarrow 6x - 9y = -15$

Eliminate x by subtraction.

$$\begin{aligned} 6x + 8y &= 36 \\ -\ (6x - 9y &= -15) \\ \hline 17y &= 51 \end{aligned}$$

$8y - (-9y)$; $36 - (-15)$

$$y = \frac{51}{17}$$

$y = 3$

Next, substitute y into one of the equations to solve for x.

$$3x + 4y = 18$$
$$3x + 4(3) = 18$$
$$3x = 18 - 12$$
$$3x = 6$$

$x = 2$

Method 2: Eliminating y

LCM of $4y$ and $3y$ = $12y$

(Multiply Equation 1 by **3** to obtain $12y$)
(Multiply Equation 2 by **4** to obtain $12y$)

$3 \times (3x + 4y = 18) \longrightarrow \mathbf{9x + 12y = 54}$
$4 \times (2x - 3y = -5) \longrightarrow \mathbf{8x - 12y = -20}$

Eliminate y by addition.

$$\begin{aligned} 9x + 12y &= 54 \\ + \ (8x - 12y &= -20) \\ \hline 17x &= 34 \\ x &= \frac{34}{17} \\ x &= 2 \end{aligned}$$

Next, substitute x into one of the equations to solve for y.

$$\begin{aligned} 3x + 4y &= 18 \\ 3(2) + 4y &= 18 \\ 4y &= 18 - 6 \\ 4y &= 12 \\ y &= 3 \end{aligned}$$

Example 5: Mary owes a total of $1500 on her credit cards. One of her credit card, MasterCard, charges 1.8%/month on her outstanding balance. While her other credit card, American Express, charges 2.1% on her balance. In one month, her total interest is $29.96. What are her balances on each of her credit cards?

First, **define** the variables.

Let m = **Balance on MasterCard**
Let a = **Balance on American Express**

Next, **set up the system of equations** by **translating the sentences**.

$$\begin{aligned} m + a &= 1500 \quad \text{(Total Balance)} \\ 0.018m + 0.021a &= 29.96 \quad \text{(Total Interest)} \end{aligned}$$

Solve for both variables using the substitution method.

Isolate m from the first equation.

$$\begin{aligned} m + a &= 1500 \\ m &= 1500 - a \end{aligned}$$

Substitute expression into m in the second equation.

$$\begin{aligned} 0.018m + 0.021a &= 29.96 \\ 0.018(1500 - a) + 0.021a &= 29.96 \\ 27 - 0.018a + 0.021a &= 29.96 \\ 0.003a &= 29.96 - 27 \\ 0.003a &= 2.96 \\ a &= \frac{2.96}{0.003} \end{aligned}$$

$a = \$986.67$

Solve for the remaining variable. Pick the easier equation of the two.

$$\begin{aligned} m + a &= 1500 \\ m + (986.67) &= 1500 \\ m &= 1500 - 986.67 \end{aligned}$$

$m = \$513.33$

The Balance of the MasterCard is $513.33.
The Balance of the American Express is $986.67

10-1 Assignment: pg. 690−691 #1, 5, 9, 13, 19, 21, 27, 31, 51; Honours #35, 55

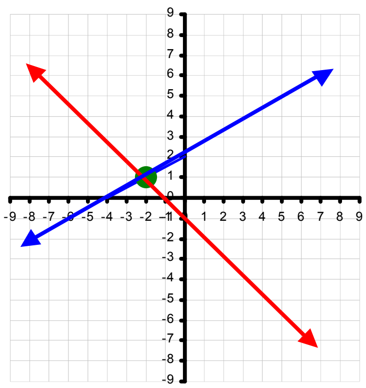
10-2 and 10-3 Systems of Equations in Two and Several Variables

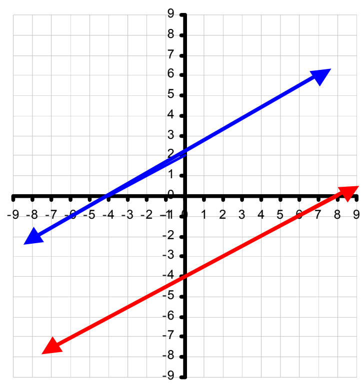
Consistent: - a system of equations having at least one solution.

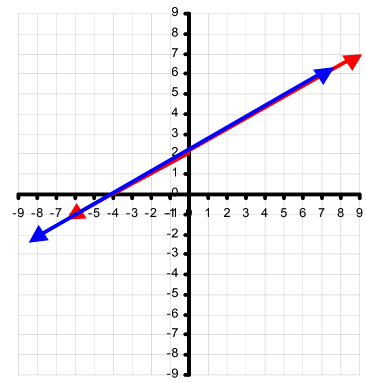
Inconsistent: - a system of equations that do ***NOT*** have a solution.

Dependent: - a system of equations having many solutions.
- must use **another variable (parameter), *t*,** to express such solution.

1. Intersecting Lines	**2. Parallel Lines**	**3. Overlapping (Coincident) Lines**
One distinct Solution	No Solution	Many (Infinite) Solutions
Different Slopes	Identical Slopes	Identical Slopes
Different y-Intercepts	Different y-Intercepts	Identical y-Intercepts







Example 1: Determine the number of solutions for the systems of equations below.

a. $x + 2y = 10$
$x + 2y = 6$

Line 1:
$x + 2y = 10$
$2y = -x + 10$
$y = \frac{-x+10}{2}$

$\mathbf{y = -\frac{1}{2}x + 5}$
$\mathbf{m = -\frac{1}{2}, y\text{-int} = 5}$

Line 2:
$x + 2y = 6$
$2y = -x + 6$
$y = \frac{-x+6}{2}$

$\mathbf{y = -\frac{1}{2}x + 3}$
$\mathbf{m = -\frac{1}{2}, y\text{-int} = 5}$

Identical slopes, but different *y*-intercepts mean parallel lines. Therefore, this system has NO SOLUTION.

Eliminate *x* by subtraction.

$$\begin{array}{rl} & x + 2y = 10 \\ - & (x + 2y = 6) \\ \hline & 0 = 4 \end{array}$$

If the system yield a *FALSE Statement* when it is solved algebraically, it is likely *a case of NO Solution*.

b. $2x + 5y = 15$
$6x + 15y = 45$

Line 1:
$2x + 5y = 15$
$5y = -2x + 15$
$y = \frac{-2x+15}{5}$

$\mathbf{y = -\frac{2}{5}x + 3}$
$\mathbf{m = -\frac{2}{5}, y\text{-int} = 3}$

Line 2:
$6x + 15y = 45$
$15y = -6x + 45$
$y = \frac{-6x+45}{15}$

$\mathbf{y = -\frac{2}{5}x + 3}$
$\mathbf{m = -\frac{2}{5}, y\text{-int} = 3}$

Identical slopes and *y*-intercepts mean overlapping lines. Therefore, this system has MANY SOLUTIONS.

Eliminate *x* by multiplication and subtraction.

$3 \times (2x + 5y = 15)$ → $6x + 15y = 45$
$1 \times (6x + 15y = 45)$ → $-\ (6x + 15y = 45)$

$$0 = 0$$

If the system yield a *TRUE Numerical Statement* when it is solved algebraically, it is likely *a case of Many Solutions*.

Let $x = t$ for any x-value of these multiple solutions (parameter). Then, $y = -\frac{2}{5}t + 3$ → $\left(t, \frac{2}{5}t + 3\right)$

Example 2: An aircraft flew from Calgary to San Francisco, a distance of 1018 km, in 2.5 hours with the tail wind. The return trip took 30 minutes longer with the head wind. Find the speed of the aircraft in still air and the speed of the wind.

First, define the variables and set up a table.

$$Speed = \frac{Distance}{Time}$$

Let x = speed of plane **Let y = speed of wind**

	Distance	Speed	Time
Tail Wind	1018 km	$x + y$	2.5 hours
Head Wind	1018 km	$x - y$	2.5 hours + 30 min = 3 hours

Next, set up the system of equations.

$$x + y = \frac{1018}{2.5} \rightarrow x + y = 407.2$$

$$x - y = \frac{1018}{3} \rightarrow x - y = 746.5\overline{3}$$

Eliminate y by addition.

$$x + y = 407.2$$
$$+ (x - y = 339.\overline{3})$$
$$2x = 746.5\overline{3}$$
$$x = \frac{746.5\overline{3}}{2}$$

$$x = 373.3 \text{ km/h}$$

Substitute x into one of the equations to solve for y.

$$x + y = 407.2$$
$$(373.3) + y = 407.2$$
$$y = 407.2 - 373.3$$

$$y = 33.9 \text{ km/h}$$

The plane was flying at 373.3 km/h.
The wind had a speed of 33.9 km/h.

Example 3: A manufacturer has a 5% vinegar solution. How much of a 40% vinegar solution can he add to bring the final concentration and volume to 20% and 875 L respectively?

Total Volume × Concentration of the Mixture = The Amount of Pure Vinegar

	Total Volume (L)	Concentration	Amount of Pure Vinegar (L)
Initial Mixture	x	0.05	$0.05x$
Amount Added	y	0.40	$0.40y$
Final Mixture	875	0.20	0.20(875)

We can form a system of equations (Total Volume and Total Amount of Pure Vinegar).

$$x + y = 875$$
$$0.05x + 0.40y = 0.20(875)$$

Eliminate x by multiplication and subtraction.

$$0.05 \times (\quad x + y = 875 \quad)$$
$$1 \times (0.05x + 0.40y = 175)$$

$$0.05x + 0.05y = 43.75$$
$$-(0.05x + 0.40y = 175)$$
$$-0.35y = -131.25$$

$$y = 375$$

$$x = 875 - y$$
$$x = 875 - (375)$$

$$x = 500$$

The manufacturer must add 375 L of 40% vinegar solution to a 500 L of 5% vinegar solution so the final mixture is 20% concentration.

In linear algebra, we can find the solutions for *n* number of variables when there are *n* number of equations relating them.

When solving systems of **3 equations with variables**:

1. **Select a set of two equations out of the three equations given where a variable can be easily eliminated.**
2. **Select another set of two equations out of the three equations given where the same variable can be eliminated (may have to use elimination by multiplication).**
3. **Once that variable is eliminated, we will be left with a system of two equations-two variables. Solve those variables.**
4. **Substitute the solutions of the two variables found in the last step into one of the three equations given originally. Find the very first variable that was eliminated.**

Example 4: Solve the system of equations $\begin{cases} 4x+5y-3z=4 \\ 5x+3y-2z=-3 \\ 3x+2y-2z=-2 \end{cases}$.

Select Equations 2 and 3 to eliminate *z*.

$$\begin{array}{r} 5x+3y-2z=-3 \\ -\ (3x+2y-2z=-2) \\ \hline 2x+y\quad\ \ =-1 \end{array}\quad \textbf{Equation 4}$$

Select Equations 1 and 2 to eliminate *z*.
(Multiply Equation 1 by **2** to obtain **−6z**)
(Multiply Equation 2 by **3** to obtain **−6z**)

$$2\times(4x+5y-3z=4) \qquad 8x+10y-6z=8$$
$$3\times(5x+3y-2z=-3) \qquad 15x+9y-6z=-9$$

$$\begin{array}{r} 8x+10y-6z=8 \\ -\ (15x+9y-6z=-9) \\ \hline -7x+y\quad\ \ =17 \end{array}\quad \textbf{Equation 5}$$

Subtract Equations 4 and 5 to eliminate *y*, and solve for *x*.

$$\begin{array}{r} 2x+y=-1 \\ -\ (-7x+y=17) \\ \hline 9x\quad\ =-18 \end{array}$$

$$x=-2$$

Substitute *x* into Equation 4 and solve for *y*.

$$2x+y=-1$$
$$2(-2)+y=-1$$
$$-4+y=-1$$
$$y=-1+4$$
$$y=3$$

Substitute *x* and *y* into Equation 3 and solve for *z*.

$$3x+2y-2z=-2$$
$$3(-2)+2(3)-2z=-2$$
$$-6+6-2z=-2$$
$$-2z=-2$$
$$z=1$$

Verify with Equation 1.

$$4x+5y-3z=4$$
$$4(-2)+5(3)-3(1)=4$$
$$-8+15-3=4$$
$$4=4$$

L.H.S = R.H.S.

Verify with Equation 2.

$$5x+3y-2z=-3$$
$$5(-2)+3(3)-2(1)=-3$$
$$-10+9-2=-3$$
$$-3=-3$$

L.H.S = R.H.S.

10-2 Assignment: pg. 697−698 #15, 17, 19, 23, 45, 49, 53; Honours #39, 57
10-3 Assignment: pg. 706−707 #17, 25, 37

