Chapter 8: Analytical Trigonometry

8-4 Inverse Trigonometric Functions

Inverse Trigonometric Function: - use when we are given a particular trigonometric ratio and we are asked to solve for the original angle measure.
- it is sometimes referred to as "*arc-sine*" ($\sin^{-1}$), "*arc-cosine*" ($\cos^{-1}$), or "*arc-tangent*" ($\tan^{-1}$).

Note: $\sin^{-1}(x) \neq \dfrac{1}{\sin(x)}$ $\quad \sin^{-1}(x) \neq (\sin x)^{-1}$ $\quad (\sin x)^{-1} = \dfrac{1}{\sin(x)} = \csc x$

Examples: For angles between 0 and $\frac{\pi}{2}$,

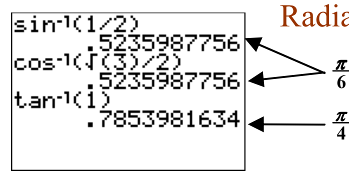
$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \rightarrow \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \rightarrow \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} \qquad \tan\left(\frac{\pi}{4}\right) = 1 \rightarrow \tan^{-1}(1) = \frac{\pi}{4}$$

To access Inverse Trigonometric Function on most calculators:

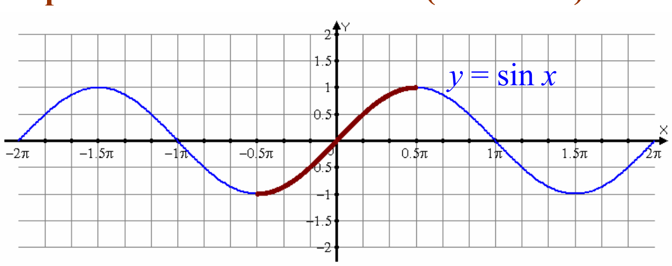
2nd **SIN** (SIN^{-1}) or **2nd** **COS** (COS^{-1}) or **2nd** **TAN** (TAN^{-1})

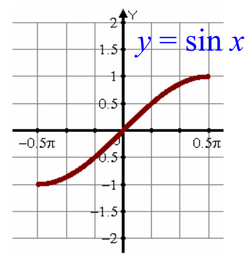
Calculator is in Radian Mode



Graph of Inverse Sine Function and its Domain and Range:

Recall that a function, $f(x)$, has to pass the vertical line test. Hence, for an **inverse function, $f^{-1}(x)$ has to pass the horizontal line test (one to one).**



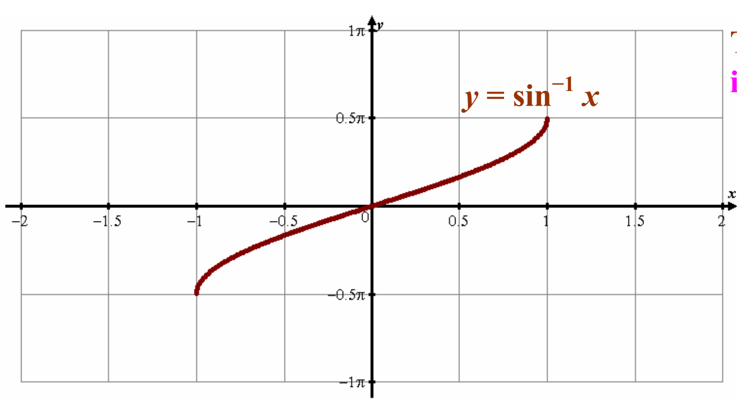


$y = \sin x$ (revised)

Domain: $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Range: $-1 \leq y \leq 1$

Note that $y = \sin x$ does NOT pass the horizontal line test between $[-2\pi, 2\pi]$. However, if we take the interval at $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, it will pass the horizontal line test. Hence, we can graph $y = \sin^{-1} x$ for y: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



Therefore, the output from a calculator of a $\sin^{-1}(x)$ input, where $-1 \leq x \leq 1$, is always between $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$y = \sin^{-1} x$

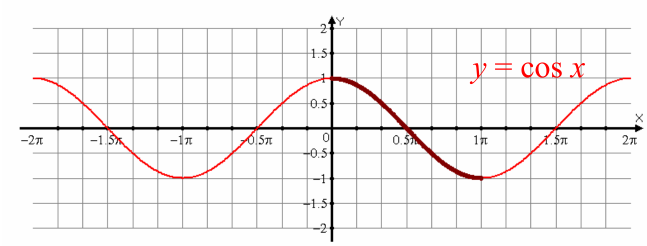
Domain: $-1 \leq x \leq 1$ $\qquad$ Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

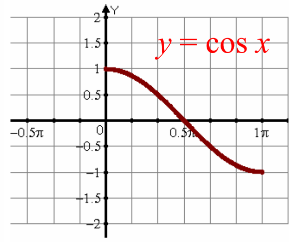
$\sin(\sin^{-1} x) = x$ $\quad$ for $-1 \leq x \leq 1$

$\sin^{-1}(\sin x) = x$ $\quad$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Graph of Inverse Cosine Function and its Domain and Range:

Recall that a function, $f(x)$, has to pass the vertical line test. Hence, for an **inverse function, $f^{-1}(x)$ has to pass the horizontal line test (one to one).**



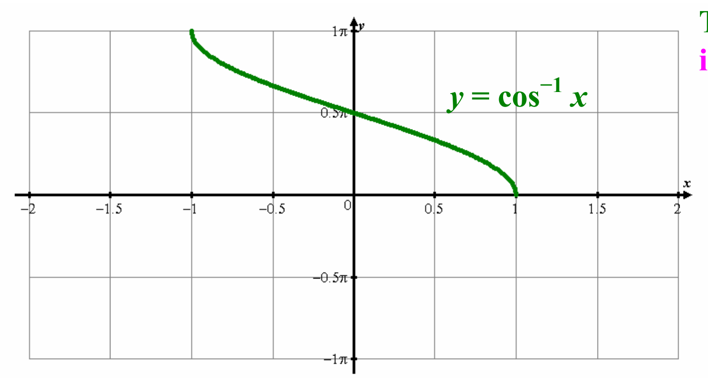


$y = \cos x$ (revised)

Domain: $0 \le x \le \pi$

Range: $-1 \le y \le 1$

Note that $y = \cos x$ does NOT pass the horizontal line test between $[-2\pi, 2\pi]$. However, if we take the interval at $[0, \pi]$, it will pass the horizontal line test. Hence, we can graph $y = \cos^{-1}x$ for y: $[0, \pi]$.



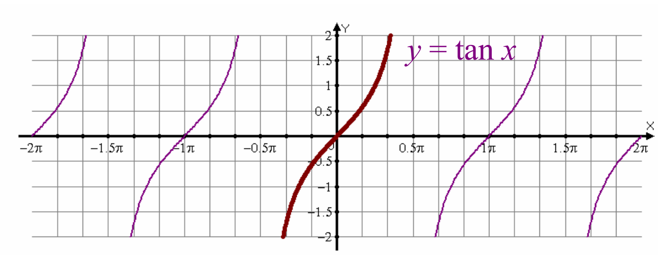
Therefore, the output from a calculator of a $\cos^{-1}(x)$ input, where $-1 \le x \le 1$, is always between $[0, \pi]$

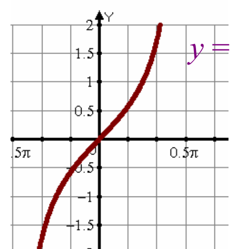
$y = \cos^{-1} x$	
Domain: $-1 \le x \le 1$	**Range: $0 \le x \le \pi$**

$\cos(\cos^{-1} x) = x$	for $-1 \le x \le 1$
$\cos^{-1}(\cos x) = x$	for $0 \le x \le \pi$

Graph of Inverse Tangent Function and its Domain and Range:

Recall that a function, $f(x)$, has to pass the vertical line test. Hence, for an **inverse function, $f^{-1}(x)$ has to pass the horizontal line test (one to one).**



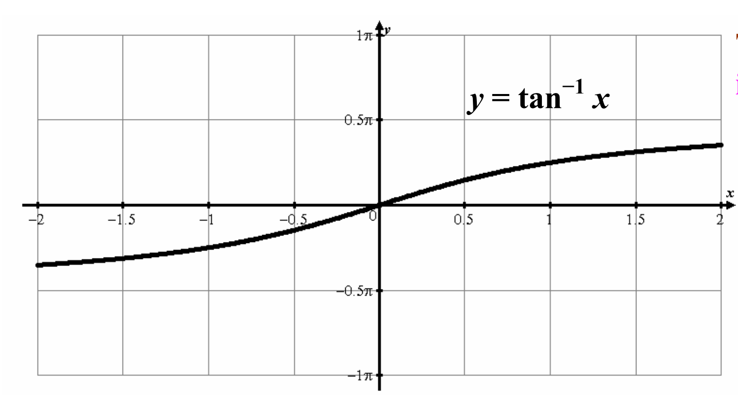


$y = \tan x$ (revised)

Domain: $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Range: $y \in R$

Note that $y = \tan x$ does NOT pass the horizontal line test between $[-2\pi, 2\pi]$. However, if we take the interval at $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, it will pass the horizontal line test. Hence, we can graph $y = \tan^{-1}x$ for y: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



Therefore, the output from a calculator of a $\sin^{-1}(x)$ input, where $-1 \le x \le 1$, is always between $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$y = \tan^{-1} x$	
Domain: $x \in R$	**Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$**

$\tan(\tan^{-1} x) = x$	for $x \in R$
$\tan^{-1}(\tan x) = x$	for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Example 1: Find the exact value of each expression, if it is defined.

a. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

From the unit circle, at $\frac{\pi}{4}$, the y-value is $\frac{\sqrt{2}}{2}$.

$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

Verify: $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\frac{\pi}{4}$ is within the range of $y = \sin^{-1}x$, which is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

b. $\tan^{-1}(-\sqrt{3})$

$\tan\left(\frac{-\pi}{3}\right) = \frac{\left(\frac{-\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} = -\sqrt{3}$

$\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

Verify: $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$ and $-\frac{\pi}{3}$ is within the range of $y = \tan^{-1}x$, which is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

c. $\cos^{-1}(3)$

No Solution

The domain of $y = \cos^{-1}x$ is $[-1, 1]$. 3 is outside this interval.

Example 2: Find the exact value of each expression, if it is defined.

a. $\sin\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)$

$= \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\sin(\sin^{-1}x) = x$ for $-1 \le x \le 1$ and $\frac{\sqrt{3}}{2}$ is within $[-1, 1]$

Calculator gives the same decimal value.

Hence, $\frac{\sqrt{3}}{2}$ is within the interval allowed.

```
sin(sin⁻¹(√(3)/2)
)
          .8660254038
√(3)/2
          .8660254038
```

b. $\sin^{-1}\left(\sin\frac{\pi}{4}\right)$

$= \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

$\sin(\sin^{-1}x) = x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and $\frac{\pi}{4}$ is within $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Calculator gives the same decimal value.

Hence, $\frac{\pi}{4}$ is within the interval allowed.

```
sin⁻¹(sin(π/4))
          .7853981634
π/4
          .7853981634
```

c. $\cos\left(\cos^{-1}\frac{\sqrt{2}}{2}\right)$

$= \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\cos(\cos^{-1}x) = x$ for $-1 \le x \le 1$ and $\frac{\sqrt{2}}{2}$ is within $[-1, 1]$

Calculator gives the same decimal value.

Hence, $\frac{\sqrt{2}}{2}$ is within the interval allowed.

```
cos(cos⁻¹(√(2)/2)
          .7071067812
√(2)/2
          .7071067812
```

d. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

$= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

$\cos^{-1}(\cos x) = x$ for $0 \le x \le \pi$ and $\frac{7\pi}{6}$ is not within $[0, \pi]$

Calculator does not give the same decimal value.

Hence, $\frac{7\pi}{6}$ is not within the interval allowed.

```
cos⁻¹(cos(7π/6))
          2.617993878
7π/6
          3.665191429
```

e. $\tan\left(\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right)$

$= \tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

$\tan(\tan^{-1}x) = x$ for $x \in R$.

Calculator gives the same decimal value.

Hence, $\frac{-\sqrt{3}}{3}$ is within the interval allowed.

```
tan(tan⁻¹(-√(3)/3
)
          -.5773502692
-√(3)/3
          -.5773502692
```

f. $\tan^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right)$

$= \tan^{-1}(-1) = -\frac{\pi}{4}$

$\tan^{-1}(\tan x) = x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and $\frac{\pi}{4}$ is within $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Calculator gives the same decimal value.

Hence, $\frac{\pi}{4}$ is within the interval allowed.

```
tan⁻¹(tan(-π/4))
          -.7853981634
-π/4
          -.7853981634
```

g. $\sin(\cos^{-1}½)$

$= \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

The domain of $y = \cos^{-1}x$ is $[-1, 1]$, and ½ is within this interval.

```
sin(cos⁻¹(1/2)
          .8660254038
√(3)/2
          .8660254038
```

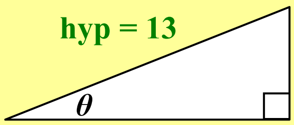
h. $\tan^{-1}\left(\sin\frac{\pi}{6}\right)$

$= \tan^{-1}(½) = $ **0.4636 rad**

The range of $y = \tan^{-1}x$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and 0.4636 rad is within this interval.

```
tan⁻¹(sin(π/6))
          .463647609
```

Example 3: Evaluate $\cos\left(\sin^{-1}\frac{5}{13}\right)$ by sketching a triangle.



Recall **SOH CAH TOA**

$\sin\theta = \frac{opp}{hyp}$ $\cos\theta = \frac{adj}{hyp}$ $\tan\theta = \frac{opp}{adj}$

When we simplify $\left(\sin^{-1}\frac{5}{13}\right)$, we are solving for the angle (θ). Therefore, we can let $\theta = \left(\sin^{-1}\frac{5}{13}\right)$.

$\cos\left(\sin^{-1}\frac{5}{13}\right) = \cos\theta$

$\cos\theta = \frac{adj}{hyp} = \frac{x}{13} = \frac{12}{13}$

Using Pythagorean Theorem:

$x^2 + 5^2 = 13^2$

$x^2 = 169 - 25$

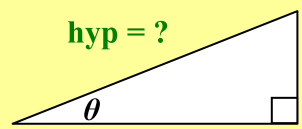
$x^2 = 144$ $x = 12$

$\cos\left(\sin^{-1}\frac{5}{13}\right) = \frac{12}{13}$

Example 4: Rewrite $\sin(\tan^{-1} x)$ as an algebraic expression in x

We can let $\theta = \tan^{-1} x$, which can be rewrite as $\tan\theta = \frac{opp}{adj} = x = \frac{x}{1}$.

Now drawing and labelling the triangle,
we can find an expression for the hypotenuse.



$(\text{hyp})^2 = x^2 + 1^2$

$\text{hyp} = \sqrt{x^2+1}$

Going back to the original expression,

$\sin(\tan^{-1} x) = \sin\theta = \frac{opp}{hyp} = \frac{x}{\sqrt{x^2+1}}$

$\sin(\tan^{-1} x) = \frac{x}{\sqrt{x^2+1}}$

8-4 Assignment: pg. 605−607 #1, 5, 11, 15, 17, 19, 21, 23, 25, 57
Honours #29, 31, 35, 43, 58

8-5 Trigonometric Equations

Some Basic Trigonometric Definitions and Identities (proven equations)

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} \qquad \cos^2\theta + \sin^2\theta = 1$$

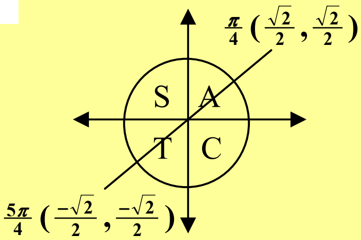
Using regular factoring technique (common factor and factoring trinomials), simplifying rational expressions, substituting with basic trigonometric definitions listed above, and referring to the unit circle, we can solve for the solution of various type of trigonometric equations.

Example 1: Find the solutions for $0 \le x < 2\pi$.

a. $\tan x - 1 = 0$

$\tan x = 1$

$x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$



b. $2\sin x\cos x = 3\sin x$

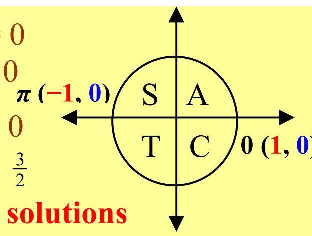
$2\sin x\cos x - 3\sin x = 0$

$\sin x\,(2\cos x - 3) = 0$

$\sin x = 0 \qquad 2\cos x - 3 = 0$

$\cos x = \frac{3}{2}$

$x = 0$ or $\pi \qquad x =$ No solutions



c. $2\cos^2 x + 3\cos x + 1 = 0$

Let $a = \cos x$

$2a^2 + 3a + 1 = 0$

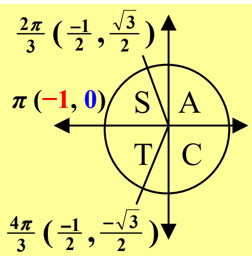
$(2a + 1)(a + 1) = 0$

$(2a + 1) = 0 \qquad (a + 1) = 0$

$a = -\frac{1}{2} \qquad a = -1$

$\cos x = -\frac{1}{2} \qquad \cos x = -1$

$x = \frac{2\pi}{3}$ or $\frac{4\pi}{3} \qquad x = \pi$



d. $\csc(3x) = 2$

Let $a = 3x \rightarrow \frac{a}{3} = x$

$0 \le x < 2\pi$ (solve x in 1 rev)

$0 \le \frac{a}{3} < 2\pi$

$0 \le a < 6\pi$ (solve a in 1 rev)

$\csc a = 2$

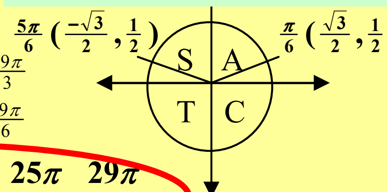
$\frac{1}{\sin a} = 2$

$\sin a = \frac{1}{2}$

$a = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{3}, \frac{25\pi}{3}, \frac{29\pi}{3}$

$3x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$

$x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$



e. $2\sin^2\left(\frac{x}{2}\right) = 1$

Let $a = \frac{x}{2}$

$2a = x$

$0 \le x < 2\pi$ (solve x in 1 rev)

$0 \le 2a < 2\pi$

$0 \le a < \pi$ (solve a in ½ rev)

$2\sin^2 a = 1$

$\sin^2 a = \frac{1}{2}$

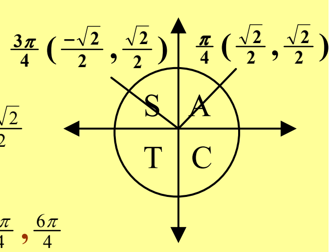
$\sin a = \pm\sqrt{\frac{1}{2}}$

$\sin a = \frac{\pm 1}{\sqrt{2}} = \frac{\pm\sqrt{2}}{2}$

$a = \frac{\pi}{4}, \frac{3\pi}{4}$

$\frac{x}{2} = \frac{\pi}{4}, \frac{3\pi}{4} \rightarrow x = \frac{2\pi}{4}, \frac{6\pi}{4}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$



f. $3 = 5\sin^2 x - 9\cos x$ (round to the 4th decimal place)

[Substitute identity: $\sin^2 x = (1 - \cos^2 x)$]

$0 = 5\sin^2 x - 9\cos x - 3$

$0 = 5(1 - \cos^2 x) - 9\cos x - 3$

$0 = 5 - 5\cos^2 x - 9\cos x - 3$

$0 = -5\cos^2 x - 9\cos x + 2$

$5\cos^2 x + 9\cos x - 2 = 0$

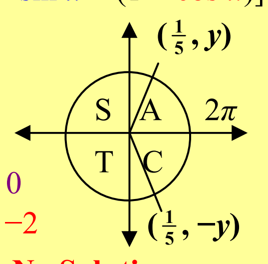
$(5\cos x - 1)(\cos x + 2) = 0$

$(5\cos x - 1) = 0 \qquad (\cos x + 2) = 0$

$\cos x = \frac{1}{5} \qquad \cos x = -2$

$\cos^{-1}\left(\frac{1}{5}\right) = x \qquad x =$ No Solution

$x = 1.3694$ rad, 4.9137 rad




```
cos⁻¹(1/5)
        1.369438406
2π-Ans
        4.913746901
```

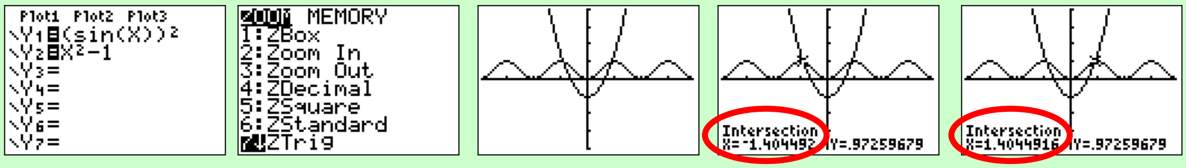
Example 2: Using a graphing calculator, determine the solutions of $\sin^2 x = x^2 - 1$.

In Radian Mode, enter each side of the equation as Y_1 and Y_2 in **Y=** screen

ZOOM Select **ZTrig**

Run Intersect twice from 2nd TRACE

$x = \pm 1.4045$



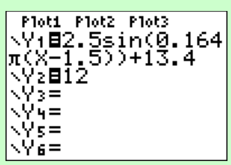
Example 3: The height of a tidal wave above the average sea level is related to time by the function, $d(t) = 2.5 \sin 0.164\pi(t - 1.5) + 13.4$, where d represents depth in metres, above sea level and t is the time in hours ($t = 0$ means midnight). When within one whole day is the tidal wave at a depth of at least 12 m for a ship to dock safely?

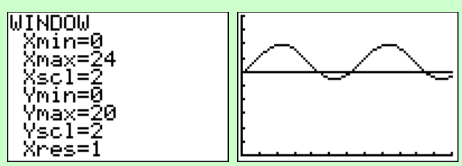
Amplitude = $|a|$ = **2.5 m**
Vert. Disp. = c = **13.4**

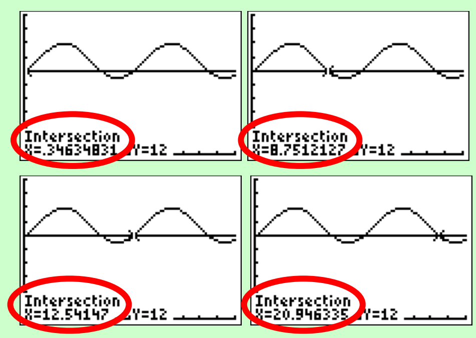
Maximum = 13.4 m + 2.5 m = 15.9 m
Minimum = 13.4 m − 2.5 m = 10.9 m

Since we are given the output (12 m) to find the input (time), we need to enter two equations and find the intersecting points.

In Radian Mode, enter the equations as Y_1 and Y_2 in **Y=** screen







Water depth will be above 12 m at

0.346 hrs < t < 8.751 hrs and 12.541 hrs < t < 20.946 hrs
12:21 AM < t < 8:45 AM and 12:33 PM < t < 8:56 PM

8-5 Assignment: pg. 616−619 #5, 11, 17, 19, 23, 31, 35, 41, 51b, 55b, 73, 79, 83; Honour #43

