Chapter 4: Polynomial and Rational Functions

4-1 Polynomial Functions and Their Graphs

Polynomial Functions: - a function that consists of a polynomial expression in a form of

$$P(x) = ax^n + bx^{(n-1)} + cx^{(n-2)} + dx^{(n-3)} + \ldots + \text{constant term} \qquad \text{where } n \in W$$

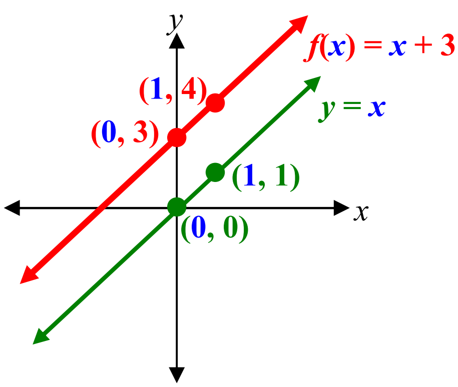
Leading Coefficient (a): - the coefficient of the largest degree term of a polynomial function.

Root: - commonly known as **solution, or zero**.
- the **value for x** when $P(x) = 0$, which is the **x-intercept** of the polynomial function **(when $y = 0$)**.

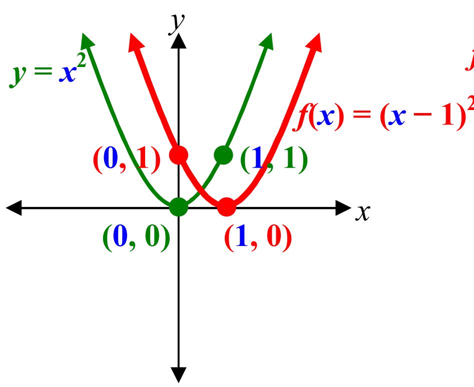
Graphs of Simple Monomials and their Transformations

Example 1: Sketch the following functions and their transformations. Label at least two points

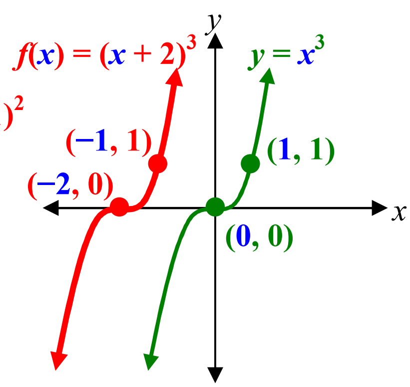
a. $y = x$ and $f(x) = x + 3$



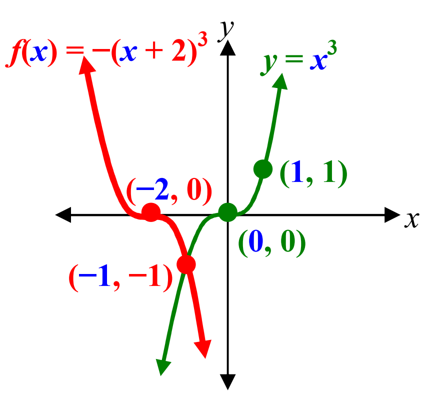
b. $y = x^2$ and $f(x) = (x - 1)^2$



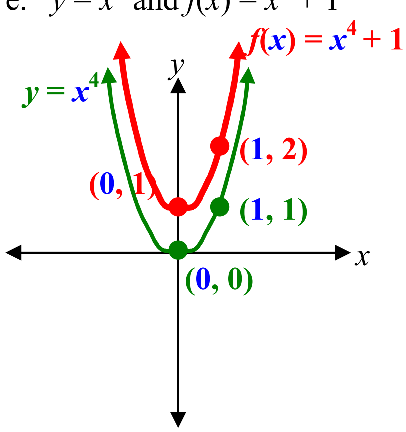
c. $y = x^3$ and $f(x) = (x + 2)^3$



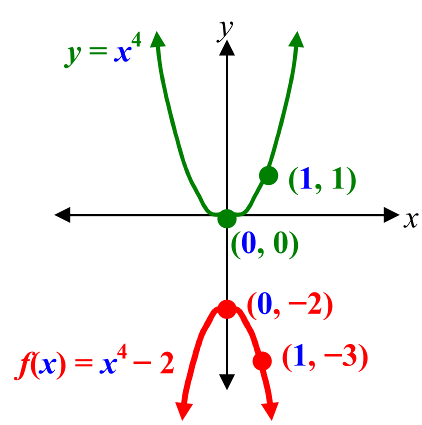
d. $y = x^3$ and $f(x) = -(x + 2)^3$



e. $y = x^4$ and $f(x) = x^4 + 1$

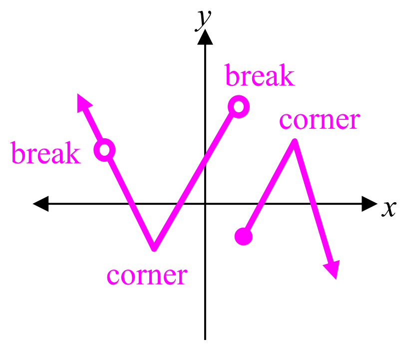


f. $y = x^4$ and $f(x) = -x^4 - 2$

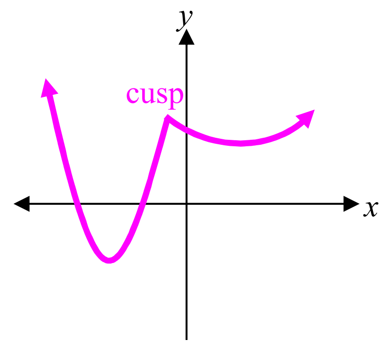


Note how the "**end behaviours**" of **Even and Odd Degree Polynomial Functions** change with the **Sign of the Leading Coefficient** (see the next page).

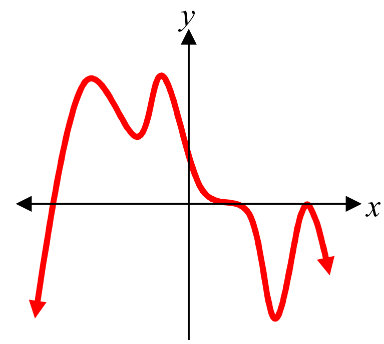
Graphs of Polynomials: - a polynomial graph is smooth and continuous.



Not a Polynomial Function
Corners and Discontinuous

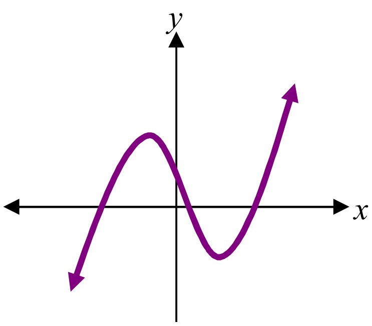


Not a Polynomial Function
Continuous but Not Smooth

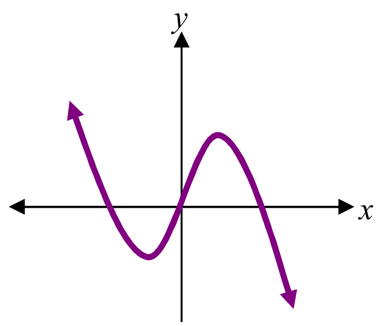


Polynomial Function
Smooth and Continuous

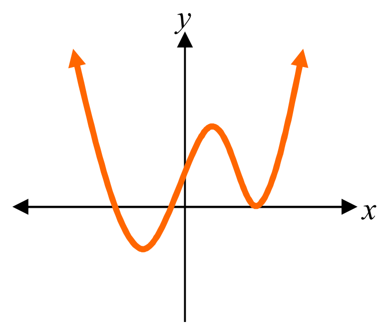
End Behaviours and Leading Terms



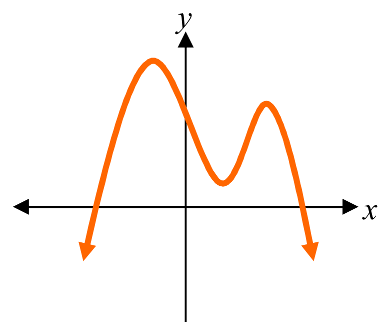
Odd Degree Polynomial Function and Positive Leading Coefficient, $a > 0$



Odd Degree Polynomial Function and Negative Leading Coefficient, $a < 0$



Even Degree Polynomial Function and Positive Leading Coefficient, $a > 0$



Even Degree Polynomial Function and Negative Leading Coefficient, $a < 0$

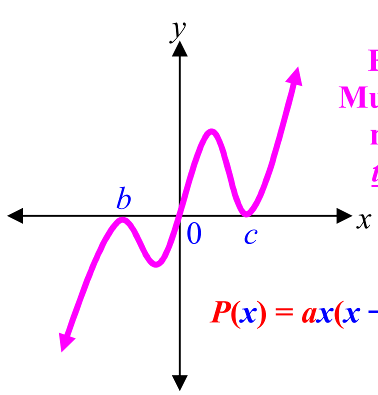
Odd Degree Polynomial Functions

When $a > 0$, Left is Downward ($y \to -\infty$ as $x \to -\infty$) and Right is Upward ($y \to \infty$ as $x \to \infty$).
When $a < 0$, Left is Upward ($y \to \infty$ as $x \to -\infty$) and Right is Downward ($y \to -\infty$ as $x \to \infty$).

Even Degree Polynomial Functions

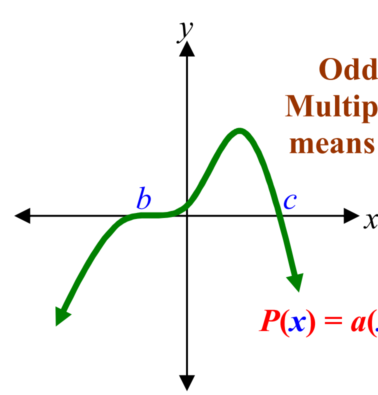
When $a > 0$, Left is Upward ($y \to \infty$ as $x \to -\infty$) and Right is Upward ($y \to \infty$ as $x \to \infty$).
When $a < 0$, Left is Downward ($y \to -\infty$ as $x \to -\infty$) and Right is Downward ($y \to -\infty$ as $x \to \infty$).

Multiplicity: - when a factored polynomial expression has exponents on the factor that is greater than 1.



Even Multiplicity or Multiplicities of 2, 4, 6… means x-intercept is *tangent to the x-axis*

$P(x) = ax(x - b)^{2,\,4,\,6\,\ldots}(x - c)^{2,\,4,\,6,\ldots}$



Odd Multiplicity or Multiplicities of 3, 5, 7… means *inflection* at x-int

$P(x) = a(x - b)^{3,\,5,\,7\,\ldots}(x - c)$

Graphing Polynomial Functions:

1. **FACTOR the Polynomial**. **Obtain the x-intercepts by letting each *Factor EQUAL to Zero and Solve***.
2. ***Label* the x-intercepts**. **Note any *Multiplicities***.
3. ***Analyze the Polynomial Function* and determine its *End Behaviours***. ***Sketch the Graph* using the *End Behaviours* as well as the x-intercepts.**
4. ***To verify, find the y-intecept by letting x = 0*. (Alternatively, the *constant term* from the expanded polynomial function *is the y-intercept*.)** If y-int = 0, let x equal to a simple number and solve for y.

Example 2: Sketch the graph of the polynomial function. Label all intercepts and explain the end behaviours.

a. $P(x) = -\frac{1}{4}x^3(x-4)^2(x+6)$

$P(x) = -\frac{1}{4}x^3(x-4)^2(x+6)$

$x = 0$, $(x-4) = 0$, $(x+6) = 0$

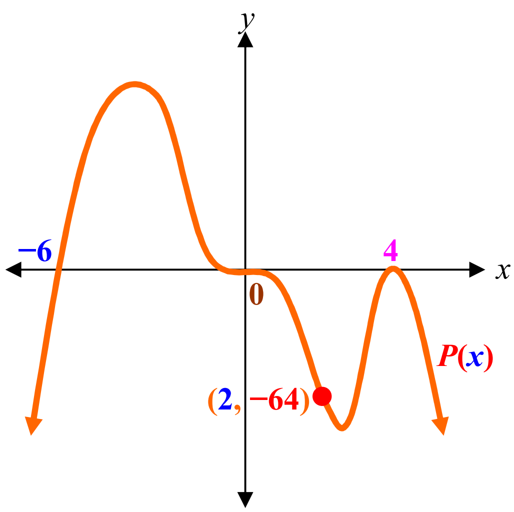
x-ints = 0 (multi-three); 4 (multi-two) and −6
inflection tangent to x-axis

$a < 0$ $(a = -\frac{1}{4})$

Overall Degree = 3 + 2 + 1 = 6 - Even

End Behaviour:

Left is Downward ($y \to -\infty$ as $x \to -\infty$) and Right is Downward ($y \to -\infty$ as $x \to \infty$).



Verify: Let $x = 2$ (since y-int = 0)

$P(2) = -\frac{1}{4}(2)^3((2)-4)^2((2)+6)$

$P(2) = -\frac{1}{4}(8)(-2)^2(8)$

$P(2) = -\frac{1}{4}(8)(4)(8)$

$P(2) = -64$

We can see that the graph we sketched has a negative y-value when $0 < x < 4$.

b. $P(x) = (x-1)(x+3)^2(x-7)^2$

$P(x) = (x-1)(x+3)^2(x-7)^2$

$(x-1) = 0$, $(x+3) = 0$, $(x-7) = 0$

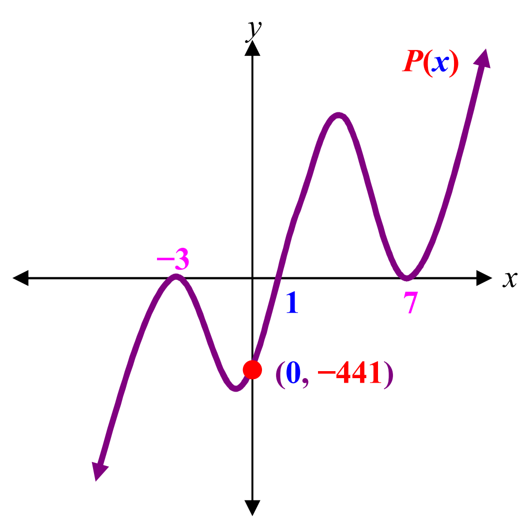
x-ints = 1 ; −3 (multi-two) and 7 (multi-two)
tangent to x-axis tangent to x-axis

$a > 0$ $(a = 1)$

Overall Degree = 1 + 2 + 2 = 5 - Odd

End Behaviour:

Left is Downward ($y \to -\infty$ as $x \to -\infty$) and Right is Upward ($y \to \infty$ as $x \to \infty$).



Verify: Let $x = 0$ for y-int

$P(0) = ((0)-1)((0)+3)^2((0)-7)^2$

$P(0) = (-1)(3)^2(-7)^2$

$P(0) = (-1)(9)(49)$

$P(0) = -441$

We can see that the graph we sketched has a negative y-value when $-3 < x < 1$.

Example 3: Factor the polynomials function, $P(x) = x^3 - 9x^2 - 4x + 36$. Without sketching the graph, describe the functions by its intercepts and explain its end behaviours.

$P(x) = x^3 - 9x^2 - 4x + 36$
$P(x) = (x^3 - 9x^2)$ ■ $(4x$ ■ $36)$ Factor by Grouping
$P(x) = x^2(x - 9) - 4(x - 9)$ GCF of each bracket
$P(x) = (x - 9)(x^2 - 4)$ Factor out common bracket
$\mathbf{P(x) = (x - 9)(x - 2)(x + 2)}$ Factor Diff of Squares

$(x - 9) = 0,\quad (x - 2) = 0,\quad (x + 2) = 0$
x-ints = 9, 2, and −2 **(No Multiplicities)**

$a > 0$ ($a = 1$) **Overall Degree = 3 – Odd**

End Behaviour:
Left is Downward ($y \to -\infty$ as $x \to -\infty$)
and Right is Upward ($y \to \infty$ as $x \to \infty$).

Let $x = 0$ for y-int
$P(0) = (0)^3 - 9(0)^3 - 4(0) + 36$ $P(0) = 36$

Example 4: Factor the polynomials functions below. Sketch their graphs by labelling all intercepts and explain their end behaviours.

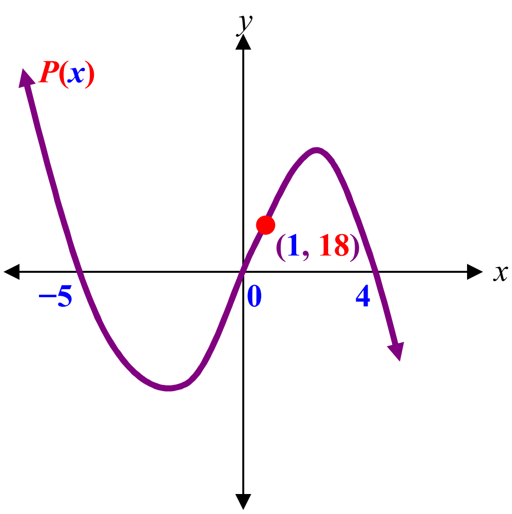
a. $P(x) = -x^3 - x^2 + 20x$

$P(x) = -x^3 - x^2 + 20x$
$P(x) = -x(x^2 + x - 20)$ Take out common factor
$\mathbf{P(x) = -x(x + 5)(x - 4)}$ Factor Trinomial

$-x = 0,\quad (x + 5) = 0,\quad (x - 4) = 0$
x-ints = 0, −5, and 4

$a < 0$ ($a = -1$) **Overall Degree = 3 - Odd**

End Behaviour:
Left is Upward ($y \to \infty$ as $x \to -\infty$)
and Right is Downward ($y \to -\infty$ as $x \to \infty$).



Verify: Let $x = 1$ (since y-int = 0)
$P(1) = -(1)^3 - (1)^2 + 20(1)$
$P(1) = -(1) - (1) + 20$
$P(1) = 18$

We can see that the graph we sketched has a positive y-value when $0 < x < 4$.

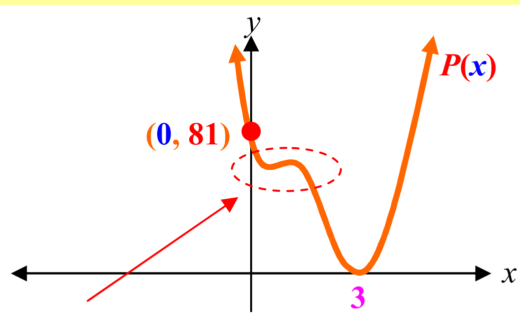
b. $P(x) = x^4 - 3x^3 - 27x + 81$

$P(x) = x^4 - 3x^3 - 27x + 81$
$P(x) = (x^4 - 3x^3)$ ■ $(27x$ ■ $81)$ Factor by Grouping
$P(x) = x^3(x - 3) - 27(x - 3)$ GCF of each bracket
$P(x) = (x - 3)(x^3 - 27)$ Factor out common bracket
$P(x) = (x - 3)(x - 3)(x^2 + 3x + 9)$ Factor Diff of Cubes

Note that the last trinomial is not factorable nor it yeilds a set of real roots using the quadratic formula (the discriminant, $b^2 - 4ac < 0$). Hence, there will not no x-ints associate with that factor (more in 4.4).

$\mathbf{P(x) = (x - 3)^2(x^2 + 3x + 9)}$ $(x - 3) = 0$
x-ints = 3 (multi-two) - tangent to x-axis
$a > 0$ ($a = 1$) **Overall Degree = 4 - Even**

End Behaviour: **Left is Upward ($y \to \infty$ as $x \to -\infty$)**
and Right is Upward ($y \to \infty$ as $x \to \infty$).



Note the bumps here indicating that there are a set of complex roots. (If the remaining trinomial has real roots, they would have come down to the x-axis and generate two x-ints.)

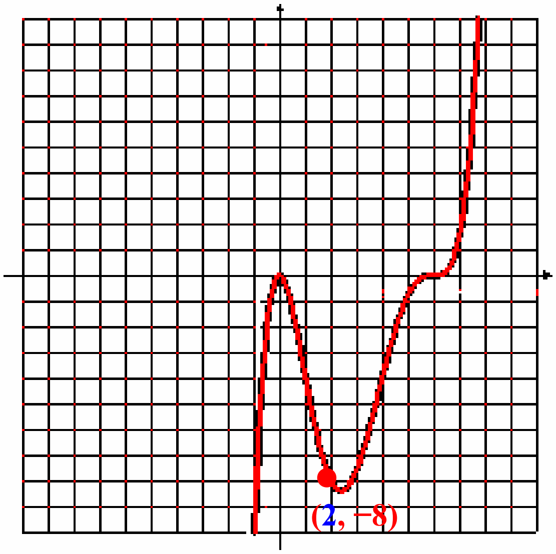
Verify: Let $x = 0$ for y-int
$P(0) = (0)^4 - 3(0)^3 - 27(0) + 81$ $P(0) = 81$

Note the constant term of a polynomial function is its y-int.

The graph we sketched has a positive y-value at any x.

Example 5: The graph below represents the polynomial function, $P(x)$ with the smallest possible degree. Write out the function in its factored form.

a.



From the graph:

x-ints = 0 (multi-two) and 6 (multi-three)
tangent to x-axis **inflection**

Hence the factors are:

$x = 0 \rightarrow (x)$ $x = 6 \rightarrow (x - 6)$

$$P(x) = ax^2(x-6)^3$$

Overall Degree = 2 + 3 = 5 - Odd

End Behaviour: **Left is Downward ($y \rightarrow -\infty$ as $x \rightarrow -\infty$) and Right is Upward ($y \rightarrow \infty$ as $x \rightarrow \infty$).**
(Therefore, we expect *a* to be positive.)

Using point **(2, −8) to solve for *a***

$-8 = a(2)^2((2) - 6)^3$

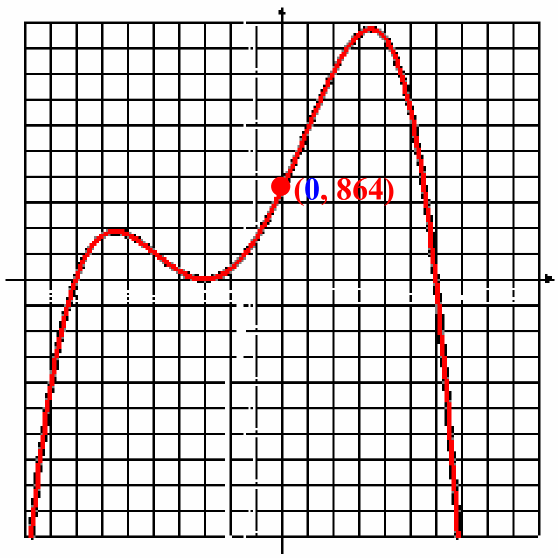
$-8 = a(4)(-4)^3$

$-8 = a(4)(-64)$

$-8 = -256a$

$\frac{-8}{-256} = a$ $a = \frac{1}{32}$ $\boxed{P(x) = \frac{1}{32}x^2(x-6)^3}$

b.



From the graph:

x-ints = −3 (multi-two) - tangent to x-axis, −8 and 6

Hence the factors are:

$x = -3 \rightarrow (x + 3)$ $x = -8 \rightarrow (x + 8)$ $x = 6 \rightarrow (x - 6)$

$$P(x) = a(x+3)^2(x+8)(x-6)$$

Overall Degree = 2 + 1 + 1 = 4 - Even

End Behaviour: **Left is Downward ($y \rightarrow -\infty$ as $x \rightarrow -\infty$) and Right is Downward ($y \rightarrow -\infty$ as $x \rightarrow \infty$).**
(Therefore, we expect *a* to be negative.)

Using the y-int at **(0, 864) to solve for *a***

$864 = a((0) + 3)^2((0) + 8)\ ((0) - 6)$

$864 = a(3)^2(8)(-6)$

$864 = a(9)(8)(-6)$

$864 = -432a$

$\frac{864}{-432} = a$ $a = -2$ $\boxed{P(x) = -2(x-3)^2(x+8)(x-6)}$

4-1 Assignment: Intro to Graphing Polynomials Worksheet and pg. 316−319 #5 to 10 (all), 13, 15, 17, 19, 23, 27, 33, 45, 77, 79;
Honours: #21 and 81

4-2 Dividing Polynomials

Consider 562 ÷ 3.

$$\begin{array}{r} 187 \\ 3\overline{)562} \\ \underline{3} \\ 26 \\ \underline{24} \\ 22 \\ \underline{21} \\ R\,1 \end{array}$$

We can say that $\dfrac{562}{3} = 187 + \dfrac{1}{3}$

(Original Number: 562; Divisor: 3; Quotient: 187; Remainder: 1; Divisor: 3)

or, we can say $562 = (3)(187) + 1$

(Original Number: 562; Divisor: (3); Quotient: (187); Remainder: 1)

Polynomial Function **Divisor Function**

In general, for $P(x) \div D(x)$, we can write

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R}{D(x)} \quad \text{or} \quad P(x) = D(x)\,Q(x) + R$$

Restriction: $D(x) \neq 0$

Quotient Function **Remainder**

Long Division to Divide Polynomials:

Example 1: Divide $\dfrac{6x^3 + 9x^2 + 15x + 21}{2x+1}$

$$\frac{6x^3 + 9x^2 + 15x + 21}{2x+1} = \frac{6x^3}{2x+1} + \frac{9x^2}{2x+1} + \frac{15x}{2x+1} + \frac{21}{2x+1}$$

You cannot divide a monomial by a polynomial!

$$\begin{array}{r} 3x^2 + 3x + 6 \\ (2x+1)\overline{)6x^3 + 9x^2 + 15x + 21} \\ \underline{-(6x^3 + 3x^2)} \\ 6x^2 + 15x \\ \underline{-(6x^2 + 3x)} \\ 12x + 21 \\ \underline{-(12x + 6)} \\ R = 15 \end{array}$$

Dividing by Polynomial is only possible by Long Division!

$$\frac{6x^3 + 9x + 15x + 21}{2x+1} = (3x^2 + 3x + 6) + \frac{15}{2x+1}$$

or

$$6x^3 + 9x^2 + 15x + 21 = (2x+1)(3x^2 + 3x + 6) + 15$$

Example 2: Divide $\frac{3x^3 - 4x^2 + 5x - 8}{x - 2}$

$$\begin{array}{r} 3x^2 + 2x + 9 \\ (x-2)\overline{)3x^3 - 4x^2 + 5x - 8} \\ \underline{-(3x^3 - 6x^2)} \\ 2x^2 + 5x \\ \underline{-(2x^2 - 4x)} \\ 9x - 8 \\ \underline{-(9x - 18)} \\ R = 10 \end{array}$$

$$\frac{3x^3 - 4x^2 + 5x - 8}{x - 2} = (3x^2 + 2x + 9) + \frac{10}{x - 2}$$

or

$$3x^3 - 4x^2 + 5x - 8 = (x - 2)(3x^2 + 2x + 9) + 10$$

Example 3: Divide $\frac{2x^3 - 7x + 6}{x - 3}$

Missing Term from Decreasing Degree!

$$\frac{2x^3 - 7x + 6}{x - 3} = \frac{2x^3 + 0x^2 - 7x + 6}{x - 3}$$

$$\begin{array}{r} 2x^2 + 6x + 11 \\ (x-3)\overline{)2x^3 + 0x^2 - 7x + 6} \\ \underline{-(2x^3 - 6x^2)} \\ 6x^2 - 7x \\ \underline{-(6x^2 - 18x)} \\ 11x + 6 \\ \underline{-(11x - 33)} \\ R = 39 \end{array}$$

$$\frac{2x^3 - 7x + 6}{x - 3} = (2x^2 + 6x + 11) + \frac{39}{x - 3}$$

or

$$2x^3 - 7x + 6 = (x - 3)(2x^2 + 6x + 11) + 39$$

Example 4: Divide $\frac{4x^3 - 8x^2 + 7x - 1}{2x^2 + 3}$

$$\frac{4x^3 - 8x^2 + 7x - 1}{2x^2 + 3} = \frac{4x^3 - 8x^2 + 7x - 1}{2x^2 + 0x + 3}$$

Missing Term from Decreasing Degree!

$$\begin{array}{r} 2x - 4 \\ (2x^2 + 0x + 3)\overline{)4x^3 - 8x^2 + 7x - 1} \\ \underline{-(4x^3 + 0x^2 + 6x)} \\ -8x^2 + x - 1 \\ \underline{-(-8x^2 + 0x - 12)} \\ R = x + 11 \end{array}$$

$$\frac{4x^3 - 8x^2 + 7x - 1}{2x^2 + 3} = (2x - 4) + \frac{x + 11}{2x^2 + 3}$$

or

$$4x^3 - 8x^2 + 7x - 1 = (2x^2 + 3)(2x - 4) + (x + 11)$$

Synthetic Division: - a simplified method to divide polynomial by a binomial linear divisor ($ax + b$).

> **Synthetic Division only works well with divisor that is in a form of $ax + b$, where Leading Coefficient of Divisor is 1 (that is $a = 1$).**

Example 5: Divide $\dfrac{3x^3 - 4x^2 + 5x - 8}{x - 2}$

Root of the Divisor

2	3	−4	5	−8
		+	+	+
		6	4	18
	3	**2**	**9**	**10**

← **Coefficients and constant of Polynomial**

← **Add** Numbers in each column

Remainder = 10

Quotient = $3x^2 + 2x + 9$

(one degree less than the original polynomial)

Example 6: Divide $\dfrac{2x^3 - 7x + 6}{x - 3}$

3	2	0	−7	6
	↓	+	+	+
		6	18	33
	2	**6**	**11**	**39**

Remainder = 39

Quotient = $2x^2 + 6x + 11$

Example 7: Divide $\dfrac{2x^3 - 3x^2 - 5x + 6}{x + 2}$

−2	2	−3	−5	6
	↓	+	+	+
		−4	14	−18
	2	**−7**	**9**	**−12**

Remainder = −12

Quotient = $2x^2 - 7x + 9$

> **4-2 Assignment: pg. 324−326 #1, 9, 11, 13, 17, 25, 31, 41, 49, 57, 59, 63, 65;**
> **Honours: #67 and 68**

4-3 Real Zeros of Polynomials

Root: - commonly known as **solution, or zero**.

- the **value for x** when **$P(x) = 0$**, which is the **x-intercept** of the polynomial function **(when $y = 0$)**.

Consider 36 ÷ 12, the quotient is 3, the remainder is 0. Hence we can say that 12 is a factor of 36.
Similarly, **if $P(x) \div (x - b)$ and the Remainder = 0, $(x - b)$ is a factor of $P(x)$, and b is a root of $P(x)$.**

If $R = 0$ when $\dfrac{P(x)}{(x-b)}$, then $(x - b)$ is a factor of $P(x)$ and $P(b) = 0$.

$$P(x) = D(x) \times Q(x)$$

$P(x)$ = Original Polynomial **$D(x)$ = Divisor (Factor)** **$Q(x)$ = Quotient**

If $R \neq 0$ when $\dfrac{P(x)}{(x-b)}$, then $(x - b)$ is NOT a factor of $P(x)$.

$$P(x) = D(x) \times Q(x) + R(x)$$

Example 1: For the polynomial function, $P(x) = x^3 - 4x^2 + 3x - 12$,

a. Determine if the divisors $(x - 4)$ and $(x + 2)$ are factors by using synthetic or long divisions. Express each dividends in a form of $P(x) = D(x) \times Q(x) + R(x)$.

b. Evaluate $P(4)$ and $P(-2)$. What is the relationship between them and the remainder determined in part a.?

a. For $P(x) \div (x - 4)$,

4	1	−4	3	−12
		4	0	12
	1	**0**	**3**	**0**

$P(x) = (x - 4)(x^2 + 3)$

b. For $P(x) \div (x + 2)$,

−2	1	−4	3	−12
		−2	12	−30
	1	**−6**	**15**	**−42**

$P(x) = (x + 2)(x^2 - 6x - 9) - 42$

b. $P(4) = (4)^3 - 4(4)^2 + 3(4) - 12$
$P(4) = (64) - 4(16) + 12 - 12$
$P(4) = 0$

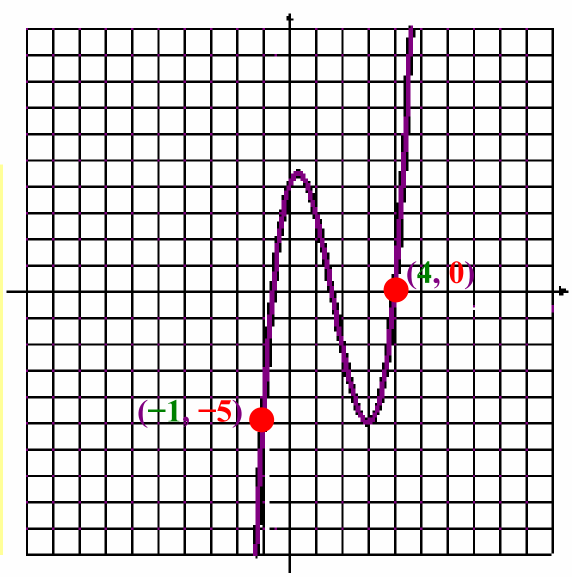
$P(-2) = (-2)^3 - 4(-2)^2 + 3(-2) - 12$
$P(-2) = (-8) - 4(4) - 6 - 12$
$P(-2) = -42$

When $P(x) \div (x - b)$, the Remainder is the same as the value of $P(b)$.

Example 2: For $P(x) = x^3 - 5x^2 + 3x + 4$, explain the relationship between

a. the order pair $(-1, -5)$, the remainder of $P(x) \div (x + 1)$ is -5, and $P(-1) = -5$.

b. the x-intercept is 4, the remainder of $P(x) \div (x - 4)$ is 0, and $P(4) = 0$.



a. For **$(-1, -5)$, it means that when $x = -1$, $y = 5$**. Hence, it has the **same meaning** as **$P(-1) = -5$**, because **$P(x) = y$**. The **remainder of $P(x) \div (x + 1)$ is -5 matches** the **y-value of the point $(-1, -5)$** as well as **$P(-1)$**.

b. For **x-intercept = 4**, the order pair is **(4, 0), it means that when $x = 4$, $y = 0$**. Hence, it has the **same meaning** as **$P(4) = 0$**, because **$P(x) = y$**. The **remainder of $P(x) \div (x - 4)$ is 0 matches** the **y-value of the point (4, 0)** as well as **$P(4)$**.

Example 3: Determine whether $x = 3$ and $x = -3i$ are roots for the function $P(x) = 4x^3 - 8x^2 - 11x - 3$.

To test if $x = 3$ is a root of $P(x)$, we evaluate $P(3)$.

$P(3) = 4(3)^3 - 8(3)^2 - 11(3) - 3$
$P(3) = 4(27) - 8(9) - 33 - 3$
$\mathbf{P(3) = 0}$

Since $P(3) = 0$, $x = 3$ is a root of $P(x)$.

To test if $x = -3i$ is a root of $P(x)$, we evaluate $P(-3i)$.

$P(-3i) = 4(-3i)^3 - 8(-3i)^2 - 11(-3i) - 3$
$P(-3i) = 4(-27i^3) - 8(9i^2) + 33i - 3$
$P(-3i) = 4(27i) - 8(-9) + 33i - 3$
$\mathbf{P(-3i) = 141i + 69 \neq 0}$

Recall, $i^2 = -1$, $i^3 = -i$

Since $P(-3i) \neq 0$, $x = -3i$ is *not* a root of $P(x)$.

The Remainder Theorem:

To find the remainder of $\frac{P(x)}{x-b}$: Substitute b from the Divisor, $(x - b)$, into the Polynomial, $P(x)$.

In general, when $\frac{P(x)}{x-b}$, $P(b)$ = Remainder.

To find the remainder of $\frac{P(x)}{ax-b}$: Substitute $\left(\frac{b}{a}\right)$ from the Divisor, $(ax - b)$, into the Polynomial, $P(x)$.

In general, when $\frac{P(x)}{ax-b}$, $P\left(\frac{b}{a}\right)$ = Remainder.

Example 4: Find the remainder of the followings using the remainder theorem.

a. $\frac{3x^3 - 4x^2 + 5x - 8}{x - 2}$ $\quad$ $x - 2 = 0$, $x = 2$

Dividing by $(x - 2)$ means substituting $x = 2$ in the numerator for the remainder.

$R = 3(2)^3 - 4(2)^2 + 5(2) - 8$
$R = 3(8) - 4(4) + 10 - 8$
$R = 24 - 16 + 10 - 8$
$\mathbf{R = 10}$

b. $\frac{2x^3 - 3x^2 - 5x + 6}{x + 3}$ $\quad$ $x + 3 = 0$, $x = -3$

Dividing by $(x + 3)$ means substituting $x = -3$ in the numerator for the remainder.

$R = 2(-3)^3 - 3(-3)^2 - 5(-3) + 6$
$R = 2(-27) - 3(9) + 15 + 6$
$R = -54 - 27 + 15 + 6$
$\mathbf{R = -60}$

c. $\frac{6x^3 - 4x^2 + 8x + 6}{2x - 3}$ $\quad$ $2x - 3 = 0$, $x = \frac{3}{2}$

Dividing by $(2x - 3)$ means substituting $x = \frac{3}{2}$ in the numerator for the remainder.

$R = 6\left(\frac{3}{2}\right)^3 - 4\left(\frac{3}{2}\right)^2 + 8\left(\frac{3}{2}\right) + 6$
$R = 6\left(\frac{27}{8}\right) - 4\left(\frac{9}{4}\right) + 12 + 6$
$R = \frac{81}{4} - 9 + 12 + 6$
$\mathbf{R = \frac{117}{4}}$

Example 5: When $P(x) = x^3 + kx^2 + 2x - 3$ is divided by $x - 4$, the remainder is -11. Find the value of k.

$P(x) \div (x - 4)$ with remainder $= -11$ means that $P(4) = -11$.

$-11 = (4)^3 + k(4)^2 + 2(4) - 3$
$-11 = (64) + k(16) + 8 - 3$
$-11 = 69 + 16k$
$-11 - 69 = 16k$
$-80 = 16k$
$\mathbf{k = -5}$

The Factor Theorem:

1. If $\frac{P(x)}{x-b}$ gives a Remainder of 0, then $(x - b)$ is the Factor of $P(x)$.

   OR

   If $P(b) = 0$, then $(x - b)$ is the Factor of $P(x)$.

2. If $\frac{P(x)}{ax-b}$ gives a Remainder of 0, then $(ax - b)$ is the Factor of $P(x)$.

   OR

   If $P\left(\frac{b}{a}\right) = 0$, then $(ax - b)$ is the Factor of $P(x)$.

Example 6: Determine whether $(x - 3)$ is a factor for the function, $P(x) = x^3 + x^2 - 9x - 9$.

To test if $(x - 3)$ is a factor of $P(x)$, we evaluate $P(3)$.

$x - 3 = 0$
$x = 3$

$P(3) = (3)^3 + (3)^2 - 9(3) - 9$
$P(3) = (27) + (9) - 27 - 9$
$P(3) = 0$

Since $P(3) = 0$, $x = 3$ is a root of $P(x)$ and $(x - 3)$ is a factor of $P(x)$.

Example 7: Determine whether $(3x - 1)$ is a factor for the function, $P(x) = 2x^3 - x^2 - 13x - 6$.

To test if $(3x - 1)$ is a factor of $P(x)$, we evaluate $P(⅓)$.

$3x - 1 = 0$
$x = ⅓$

$P(⅓) = 2(⅓)^3 - (⅓)^2 - 13(⅓) - 6$
$P(⅓) = 2\left(\frac{1}{27}\right) - \left(\frac{1}{9}\right) - \frac{13}{3} - 6$
$P(⅓) = -\frac{280}{27} \neq 0$

Since $P(⅓) \neq 0$, $x = ⅓$ is *not* a root of $P(x)$ and $(3x - 1)$ is *not* a factor of $P(x)$.

Example 8: If $P(x) = x^3 + kx^2 + kx + 21$ and 3 is a root of $P(x)$, find the value of k.

If $x = 3$ is a root of $P(x)$, it means that when $P(x) \div (x - 3)$, the remainder $= 0$ and $P(3) = 0$.

$0 = (3)^3 + k(3)^2 + k(3) + 21$
$0 = (27) + k(9) + 3k + 21$
$0 = 48 + 12k$
$0 - 48 = 12k$
$-48 = 12k$
$k = -4$

Rational Roots: - any roots of a polynomial function that belong in a set of rational numbers (any numbers that can be expressed as a fractions of integers).

Rational Roots Theorem:

For a polynomial $P(x)$, a List of POTENTIAL Rational Roots can be generated by Dividing ALL the Factors of its Constant Term by ALL the Factors of its Leading Coefficient.

$$\text{Potential Rational Zeros of } P(x) = \frac{\text{ALL Factors of the Constant Term}}{\text{ALL Factors of the Leading Coefficient}}$$

Example 9: List the potential rational roots for the following polynomials

a. $P(x) = x^3 + x^2 - 16x - 20$

Potential Rational Zeros = $\dfrac{\text{Factors of Constant Term}}{\text{Factors of Leading Coefficient}}$

Potential Rational Zeros = $\dfrac{\pm 1, 2, 4, 5, 10, 20}{\pm 1}$

Potential Rational Zeros = ±1, 2, 4, 5, 10, 20

b. $P(x) = 8x^3 + 33x^2 - 37x - 18$

Potential Rational Zeros = $\dfrac{\pm 1, 2, 3, 6, 9, 18}{\pm 1, 2, 4, 8}$

= ±1, 2, 3, 6, 9, 18, ½, $\frac{2}{2}$, $\frac{3}{2}$, $\frac{6}{2}$, $\frac{9}{2}$, $\frac{18}{2}$, ¼, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{6}{4}$, $\frac{9}{4}$, $\frac{18}{4}$, ⅛, $\frac{2}{8}$, $\frac{3}{8}$, $\frac{6}{8}$, $\frac{9}{8}$, $\frac{18}{8}$

Possible Zeros = ±1, 2, 3, 6, 9, 18, ½, $\frac{3}{2}$, $\frac{9}{2}$, ¼, $\frac{3}{4}$, $\frac{9}{4}$, ⅛, $\frac{3}{8}$, $\frac{9}{8}$

To Factor Third Degree Polynomials:

1. **Generate a List of Potential Rational Zeros** from the polynomial function, $P(x)$.
2. ***Pick some Potential Integral Zeroes (Integers type zeros)*** **and use the Factor Theorem to Test them for an actual root** of the polynomial function.
3. **Using *Synthetic Division*, find the Quotient by dividing the polynomial with the root** found in the previous step.
4. ***Find the Remaining Factors or Roots* from the quotient** (usually a quadratic function) either **by using regular factoring methods or the quadratic formula**.

Example 10: Completely factor the following polynomial functions and graph.

a. $P(x) = x^3 + 5x^2 + 2x - 8$

Potential Rational Zeros = ±1, 2, 4, 8

Test Potential Rational Zeros for Actual Roots.
$P(1) = (1)^3 + 5(1)^2 + 2(1) - 8 = \mathbf{0}$ $\therefore$ **$(x - 1)$ is a factor**

Use **Synthetic Division** to find the **Quotient**.

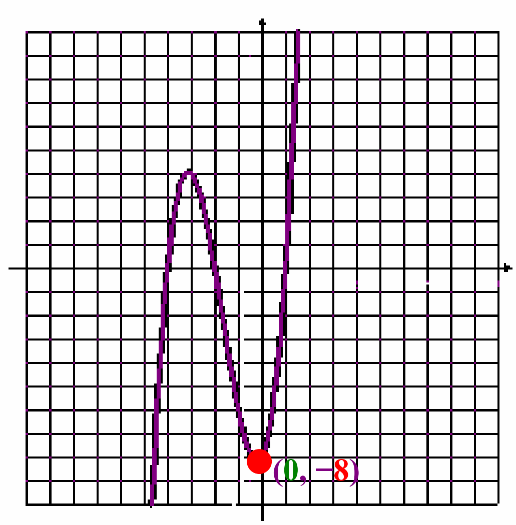
1	1	5	2	−8
		1	6	8
	1	**6**	**8**	**0**

$\mathbf{P(x) = (x - 1)(x^2 + 6x + 8)}$

Factor the Quadratic (2nd degree) Quotient and Graph.

$\mathbf{P(x) = (x - 1)(x + 2)(x + 4)}$

***y*-intercept = −8**
***x*-ints = 1, −2 and −4**



b. $P(x) = 5x^3 - 7x^2 - x + 3$

Potential Rational Zeros = ±1, 3, $\frac{1}{5}$, $\frac{3}{5}$

Test Potential Rational Zeros for Actual Roots.
$P(1) = 5(1)^3 - 7(1)^2 - (1) + 3 = \mathbf{0}$ $\therefore$ **$(x - 1)$ is a factor**

Use **Synthetic Division** to find the **Quotient**.

1	5	−7	−1	3
		5	−2	−3
	5	**−2**	**−3**	**0**

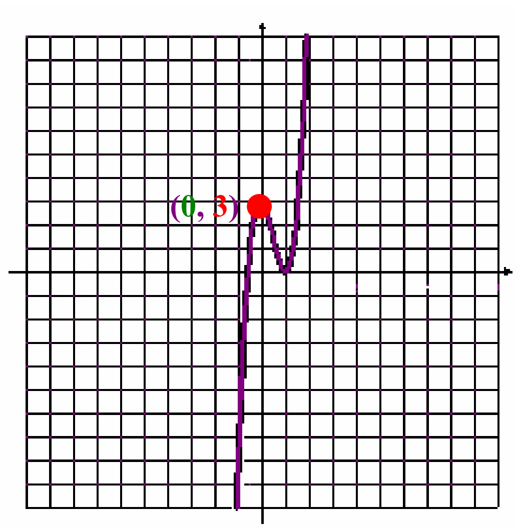
$\mathbf{P(x) = (x - 1)(5x^2 - 2x - 3)}$

Factor the Quadratic (2nd degree) Quotient and Graph.

$P(x) = (x - 1)(x - 1)(5x + 3)$

$\mathbf{P(x) = (x - 1)^2(5x + 3)}$

***y*-intercept = 3**
***x*-ints = $-\frac{3}{5}$ and 1 (Multi-of-twos)**



To Factor Higher Degree Polynomials (more than 3rd degree):

1. **Generate a List of Potential Rational Zeros** from the polynomial function, $P(x)$.
2. ***Pick some Potential Integral Zeroes (Integers type zeros)*** **and use the Factor Theorem to Test them for an actual root** of the polynomial function.
3. **Using *Synthetic Division*, find the Quotient by dividing the polynomial with the root** found in the previous step.
4. ***Find the Remaining Factors or Roots* from the quotient** (usually a function with degree one less than the original polynomial) **by repeating steps 1 through 3 until a quadratic quotient appears**. Then factor it **by using regular factoring methods or the quadratic formula**.

Example 11: Completely factor the polynomial function, $P(x) = 3x^4 + 5x^3 - 3x^2 - 9x - 4$ and graph.

Potential Rational Zeros $= \pm 1, 2, 4, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}$

Test Potential Rational Zeros for Actual Roots.

$P(1) = 3(1)^4 + 5(1)^3 - 3(1)^2 - 9(1) - 4 = -8 \neq 0$ $\therefore$ **1 is not a root**

$P(-1) = 3(-1)^4 + 5(-1)^3 - 3(-1)^2 - 9(-1) - 4 = \mathbf{0}$

$\therefore$ **$(x + 1)$ is a factor**

Use **Synthetic Division** to find the **Quotient**.

−1	3	5	−3	−9	−4
		−3	−2	5	4
	3	**2**	**−5**	**−4**	**0**

$P(x) = (x + 1)(3x^3 + 2x^2 - 5x - 4)$

Since the Quotient is a 3rd degree polynomial, we have to repeat the entire process again until we get to a quotient that is a 2nd degree.

Potential Rational Zeros of the Quotient $= \pm 1, 2, 4, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}$

Test Potential Rational Zeros for Actual Roots.

$P(1) = 3(1)^3 + 2(1)^2 - 5(1) - 4 = -4 \neq 0$ $\therefore$ **1 is not a root**

$P(-1) = 3(-1)^3 + 2(-1)^2 - 5(-1) - 4 = \mathbf{0}$ $\therefore$ **$(x + 1)$ is a factor**

Use **Synthetic Division** to find the **Quotient**.

−1	3	2	−5	−4
		−3	1	4
	3	**−1**	**−4**	**0**

$P(x) = (x + 1)(x + 1)(3x^2 - x - 4)$

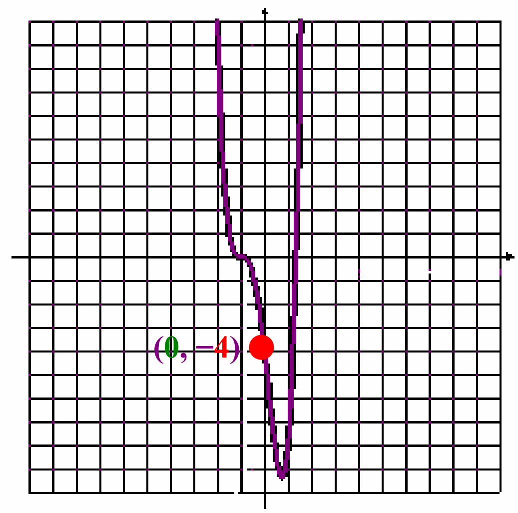
Factor the Quadratic (2nd degree) Quotient and Graph.

$P(x) = (x + 1)(x + 1)(x + 1)(3x - 4)$

$\mathbf{P(x) = (x + 1)^3(3x - 4)}$

y-intercept = −4

x-ints = $\frac{4}{3}$ and −1 (Multi-of-threes)



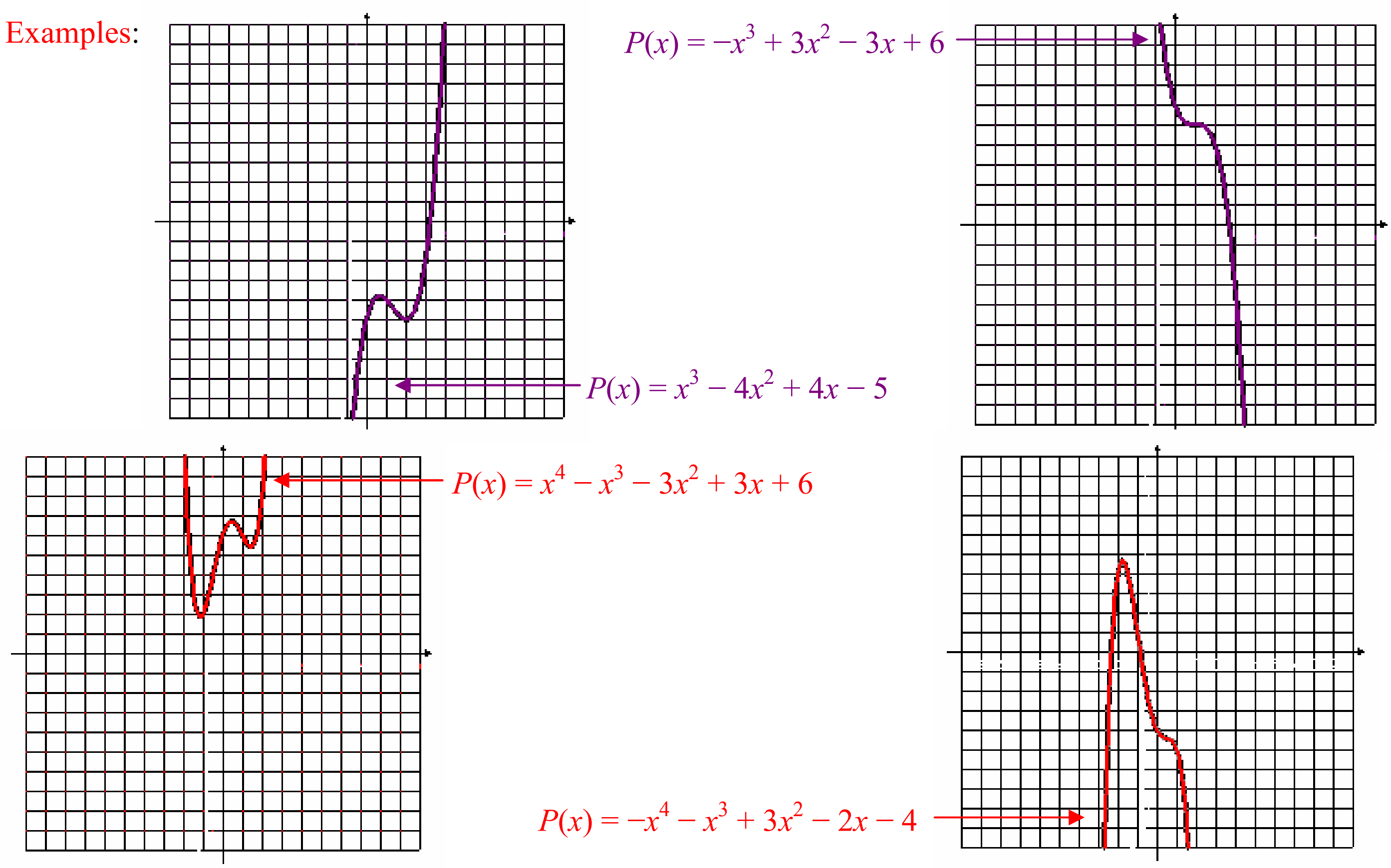
4-3 Assignment: pg. 333−336 # 7, 9, 15, 19, 29, 41, 53, 55, 79, 85, 93, 95 (you may use your calculator for this); pg 135 # 55, 65, 69, 85

Honours: pg. 333−336 #35, 45 and 100

4-4 Complex Zeros

Complex Zeros: - any roots that are not real numbers.
- it occurs when a quadratic quotient has a **negative discriminant** ($b^2 - 4ac$) **in the quadratic formula**.
- it can also occur when *none of the potential rational zeros of an even degree polynomial turn out to be the actual x-intercept of the polynomial*.
- graphs of polynomials with complex zeros ***can be recognized by their local / absolute extremes (bumps) that never lower or rise to meet the x-axis***.

Examples:



Note: **All odd degree polynomials have at least one real zero (the line must be able to go through the *x*-axis).**
Even degree polynomials can have all or some complex zeros.
In general, there are (n − 1) "bumps" for an n^{th} degree polynomial (an inflection counts as two bumps).

> **The Zero Theorem**
>
> **There are n number of solutions (complex, real or both) for any n^{th} degree polynomial function accounting that that a zero with multiplicity of k is counted k times.**

Example 1: Find the roots of the 5th degree polynomial function, $P(x) = (x + 3)(2x^2 + 5x - 3)^2$. State the multiplicity of each root.

We check if the Quadratic Factor is factorable.

$P(x) = (x + 3)[2x^2 + 5x - 3]^2$

$P(x) = (x + 3)[(x + 3)(2x - 1)]^2$

$P(x) = (x + 3)(x + 3)^2(2x - 1)^2$ $\quad$ $\mathbf{P(x) = (x + 3)^3(2x - 1)^2}$

***x*-ints = −3 (Multi-of-threes) and ½ (Multi-of-twos)**

Example 2: Write a polynomial function with −5 as a root of multiplicity of 2, and 1 as a root of multiplicity of 3, along with 0 as a root.

***x*-intercepts = −5 (Multiplicity-of-Two), 1 (Multiplicity-of-Three), and 0.**

$$P(x) = a \bullet (x+5)^2 \bullet (x-1)^3 \bullet (x-0)$$

$$P(x) = a\,x\,(x+5)^2\,(x-1)^3$$

Note that we must add, *a*, as leading coefficient because we do not know if the graph has been stretched or reflected vertically.

<u>Some Irrational Real Roots and All Complex Roots exist as Conjugate Pairs</u>.

If the irrational in a form of $a + c\sqrt{b}$ is a root, then $a - c\sqrt{b}$ is also a root.
If bi is a root, then $-bi$ is also a root.
If $a + bi$ is a root, then $a - bi$ is also a root.

Example 3: Given a 7th degree polynomial with some of the zeros equal to 3, $4 - 3i$, $2i$, and $1 + 2\sqrt{5}$, find its other zeros.

Zeros: $x = 3$ (real rational zero); $x = 4 - 3i$ **(complex zero)** → $x = 4 + 3i$
$x = 2i$ **(complex zero)** → $x = -2i$
$x = 1 + 2\sqrt{5}$ **(real irrational zero)** → $x = 1 - 2\sqrt{5}$

<u>Multiplying Factors with Conjugate Irrational Number Pairs</u>:

1. Set up the factors from the roots. *<u>Be careful with the signs</u>*. If the roots are $x = (a + c\sqrt{b})$ and $x = (a - c\sqrt{b})$, then the factors are **(*x* [redacted] $(a + c\sqrt{b})$) and (*x* [redacted] $(a - c\sqrt{b})$).**
2. **<u>The *Products* of these *Two Irrational Factors* will yield a *Quadratic Expression*</u>** $x^2 + dx + e$ where $d = -2a$ and $e = (a^2 - c^2 b)$. Basically, the ***<u>Coefficient of the Linear Term</u>* <u>is the</u> *<u>Negative Sum</u>* <u>of these irrational roots</u>**; the ***<u>Constant Term</u>* <u>is the</u> *<u>Product</u>* <u>of these roots</u>**.

<u>Multiplying Factors with Conjugate Complex Number Pairs</u>:

1. Set up the factors from the roots. *<u>Be careful with the signs</u>*. If the roots are $x = (a + bi)$ and $x = (a - bi)$, then the factors are **(*x* [redacted] $(a + bi)$) and (*x* [redacted] $(a - bi)$).**
2. **<u>The *Products* of these *Two Complex Factors* will yield a *Quadratic Expression*</u>** $x^2 + dx + e$ where $d = -2a$ and $e = (a^2 + b^2)$. Basically, the ***<u>Coefficient of the Linear Term</u>* <u>is the</u> *<u>Negative Sum</u>* <u>of these complex roots</u>**; the ***<u>Constant Term</u>* <u>is the</u> *<u>Product</u>* <u>of these roots</u>**.

Example 4: A quadratic polynomial function has a root $3 + 5i$. Determine the other root and the equation of this polynomial function.

Zeros: $x = 3 + 5i$ **(complex zero)** → $x = 3 - 5i$

$P(x) = $ (*x* [redacted] $(3 + 5i)$) (*x* [redacted] $(3 - 5i)$)

The Long Way: Multiply them out using FOIL.

$P(x) = (x - (3 + 5i))\,(x - (3 - 5i))$
$P(x) = (x - 3 - 5i)\,(x - 3 + 5i)$
$P(x) = x^2 - 3x \cancel{+ 5ix} - 3x + 9 \cancel{- 15i} \cancel{- 5ix} \cancel{+ 15i} - 25i^2$
$P(x) = x^2 - 3x - 3x + 9 - 25(-1)$

$$P(x) = x^2 - 6x + 34$$

The Short Way: $P(x) = x^2 - dx + e$

Coefficient of Linear Term, d = *Neg Sum* of roots
$d = -[(3 + 5i) + (3 + 5i)]$ $\quad d = -6$

Constant Term, e = *Product* of roots
$e = (3 + 5i)(3 - 5i)$
$e = 9 \cancel{- 15i} \cancel{+ 15i} - 25i^2$
$e = 9 - 25(-1)$ $\quad e = 34$

Example 5: Find all roots of $P(x) = x^4 - 5x^3 + 8x^2 - 20x + 16$ has a root of $2i$.

Zeros: $x = 2i$ **(complex zero)** $\rightarrow$ $x = -2i$

$P(x) = (x$ [redacted] $2i)(x$ [redacted] $(-2i)) \bullet Q(x)$ → **The Short Way:** $(x - 2i)(x + 2i) = x^2 - dx + e$

$P(x) = (x^2 + 0x + 4) \bullet Q(x)$

Coefficient of Linear Term, d = *Neg Sum* of roots

$d = -[(2i) + (-2i)]$ $\quad$ $d = 0$

Constant Term, e = *Product* of roots

$e = (2i)(-2i) = -4i^2 = -4(-1)$ $\quad$ $e = 4$

To find $Q(x)$, we can do Long Division.

$$\begin{array}{r} x^2 - 5x + 4 \\ (x^2 + 0x + 4)\overline{)x^4 - 5x^3 + 8x^2 - 20x + 16} \\ -(x^4 + 0x^3 + 4x^2) \\ \hline -5x^3 + 4x^2 - 20x \\ -(-5x^3 - 0x^2 - 20x) \\ \hline 4x^2 + 0x + 16 \\ -(4x^2 + 0x + 16) \\ \hline R = 0 \end{array}$$

$P(x) = (x^2 + 4)(x^2 - 5x + 4)$

$$P(x) = (x^2 + 4)(x - 4)(x + 1)$$

Example 6: Find a lowest degree polynomial function that has $2 - \sqrt{3}$ and $3 + 2i$ as roots.

Zeros: $x = 2 - \sqrt{3}$ **(irrational zero)** $\rightarrow$ $x = 2 + \sqrt{3}$

For $(x$ [redacted] $(2 - \sqrt{3}))(x$ [redacted] $(2 + \sqrt{3}))$

Coefficient of Linear Term, d = *Neg Sum* of roots

$d = -[(2 - \sqrt{3}) + (2 + \sqrt{3})]$ $\quad$ $d = -4$

Constant Term, e = *Product* of roots

$e = (2 - \sqrt{3})(2 + \sqrt{3})$

$e = 4 + 2\sqrt{3} - 2\sqrt{3} - (\sqrt{3})^2$

$e = 4 - 3$ $\quad$ $e = 1$

$(x - (2 - \sqrt{3}))(x - (2 + \sqrt{3})) = (x^2 - 4x + 1)$

$x = 3 + 2i$ **(complex zero)** $\rightarrow$ $x = 3 - 2i$

For $(x$ [redacted] $(3 + 2i))(x$ [redacted] $(3 - 2i))$

Coefficient of Linear Term, d = *Neg Sum* of roots

$d = -[(3 + 2i) + (3 - 2i)]$ $\quad$ $d = -6$

Constant Term, e = *Product* of roots

$e = (3 + 2i)(3 - 2i)$

$e = 9 \cancel{- 6i} \cancel{+ 6i} - 4i^2$

$e = 9 - 4(-1)$ $\quad$ $e = 13$

$(x - (3 + 2i))(x - (3 - 2i)) = (x^2 - 6x + 13)$

$P(x) = (x - (2 - \sqrt{3}))(x - (2 + \sqrt{3}))(x - (3 + 2i))(x - (3 - 2i))$

$$P(x) = (x^2 - 4x + 1)(x^2 - 6x + 13)$$

Example 7: Find all the zeros of the polynomial function, $P(x) = -2x^3 + 7x^2 - 10x + 8$.

Potential Rational Zeros = ±1, 2, 4, 8

Test Potential Rational Zeros for Actual Roots.

$P(1) = -2(1)^3 + 7(1)^2 - 10(1) + 8 \neq 0$

$P(2) = -2(2)^3 + 7(2)^2 - 10(2) + 8 = 0$ $\quad \therefore$ **$(x - 2)$ is a factor**

Use Synthetic Division to find the Quotient.

2	−2	7	−10	8
		−4	6	−8
	−2	**3**	**−4**	**0**

$P(x) = (x - 2)(-2x^2 + 3x - 4)$

The Quadratic (2nd degree) Quotient cannot be factored. We need to use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(3) \pm \sqrt{(3)^2 - 4(-2)(-4)}}{2(-2)}$$

$$x = \frac{-3 \pm \sqrt{-23}}{-4} = \frac{3 \pm i\sqrt{23}}{4}$$

$$\text{Zeros} = 2,\ \frac{3 + i\sqrt{23}}{4},\ \text{and}\ \frac{3 - i\sqrt{23}}{4}$$

4-4 Assignment: pg. 346−347 #13, 17, 25, 29, 31, 33, 37, 43, 47
Honours: pg. 346−347 #39, 49, 55 and 67

