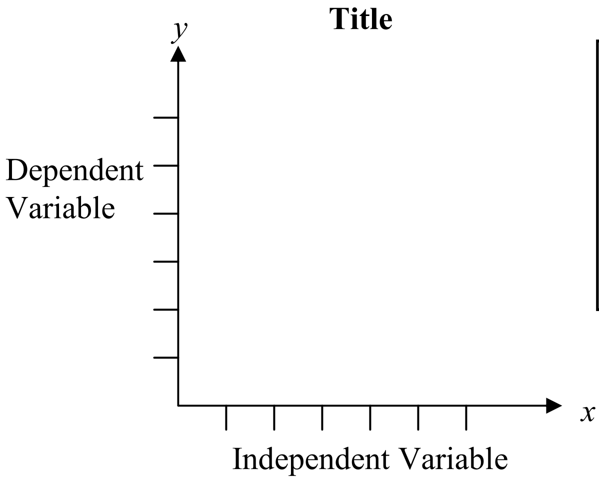
Chapter 3: Functions

3-1 What is a Function?

Relation: - an equation that explains how one variable (input x) can turn into another variable (output y).

Independent (Manipulated) Variable: - a variable that you change in a situation to cause an effect.
- label on the x-axis (horizontal axis).

Dependent (Responding) Variable: - a variable that you measure because of the changes you caused with the manipulated variable.
- label on the y-axis (vertical axis).

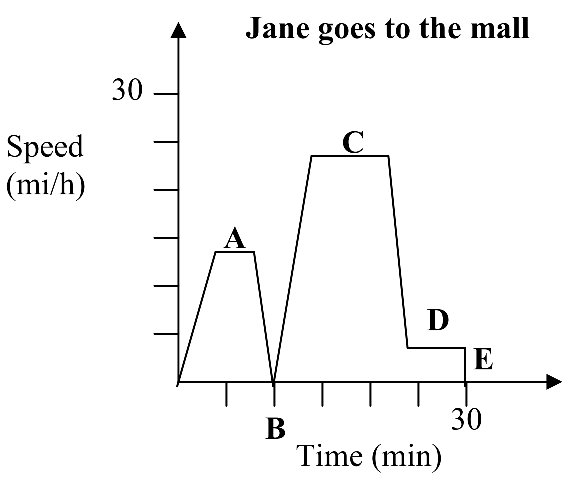


For any graph, there should be proper labeling on:

a. Title (usually y vs. x)
b. The name of the variables on the axis and their units.
c. Proper intervals scaling.
d. If there are two overlapping graphs on the same grid, a clear legend must be indicated.

Creating a Scenario to Match a Graph

Example 1: Jane drives to the shopping mall from her house. Using the graph, write a scenario that would describe her travel.



A. Jane Left her house and drove at 15 mi/h (playground zone) for the first 10 minutes.

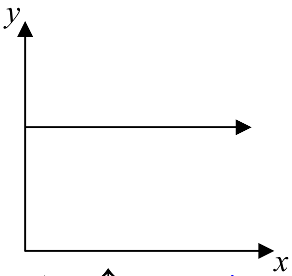
B. She stopped at the stop sign briefly.

C. Jane drove at 25 mi/h (residential zone) for 15 minutes.

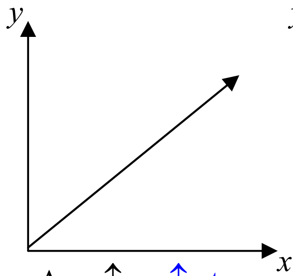
D. She entered the parking lot of the mall and spent 5 minutes looking for a parking spot (5 mi/h).

E. Jane parked and stopped.

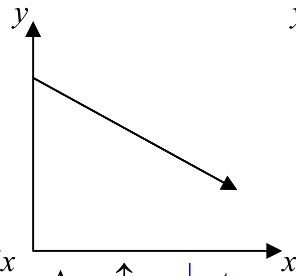
Common Shapes of Different Graphs



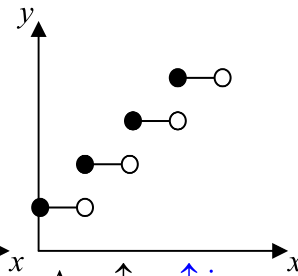
As $x\uparrow$, y remains constant.



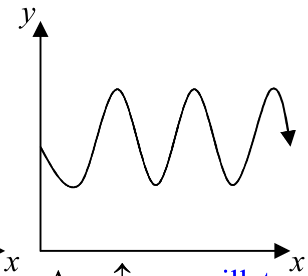
As $x\uparrow$, $y\uparrow$ at a steady rate.



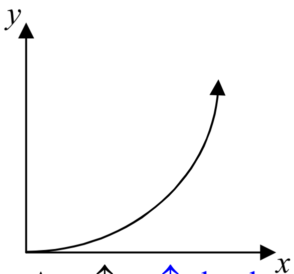
As $x\uparrow$, $y\downarrow$ at a steadv rate.



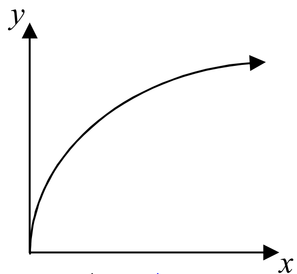
As $x\uparrow$, $y\uparrow$ in a stepwise manner.



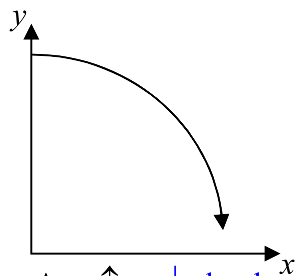
As $x\uparrow$, y oscillates back and forth.



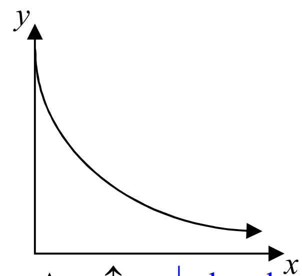
As $x\uparrow$, $y\uparrow$ slowly at first, but the faster later.



As $x\uparrow$, $y\uparrow$ sharply at first, but eventually levels off.



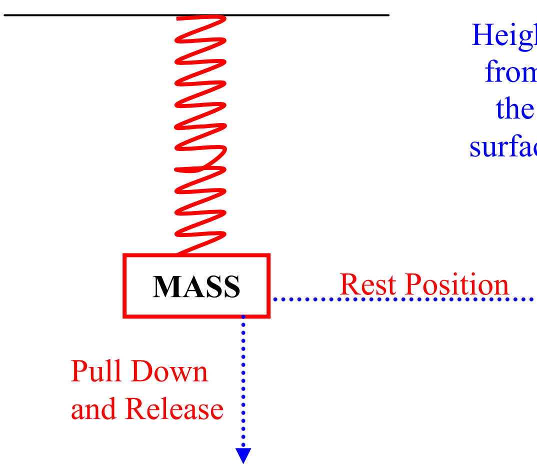
As $x\uparrow$, $y\downarrow$ slowly first, then faster later.

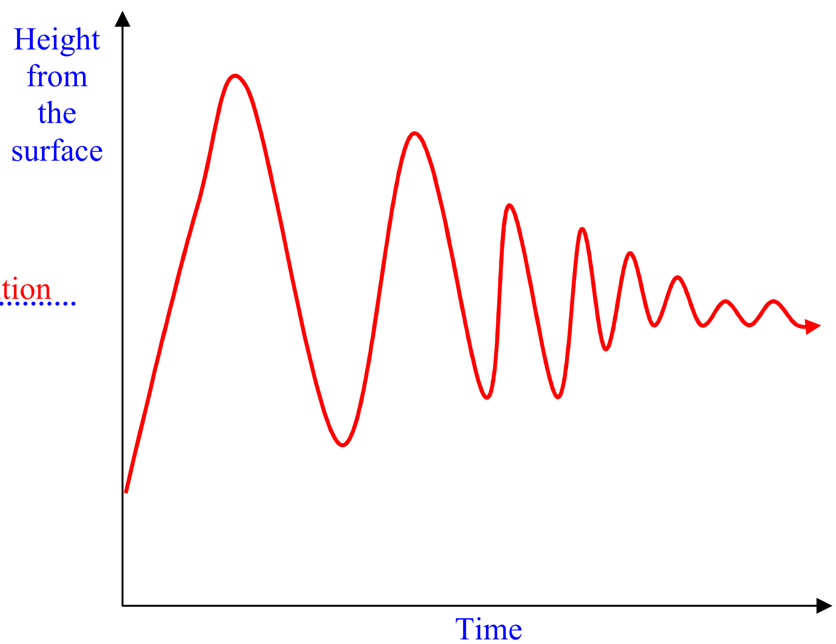


As $x\uparrow$, $y\downarrow$ sharply first, then levels off.

Creating a Graph to Illustrate a Scenario

Example 2: Imagine a mass attached to a spring. It occupies a position at rest above a level surface. If the mass is pulled down and then released, it will move up and down. Sketch the graph to represent the relationship between the height of the mass above the surface and the time after its release.



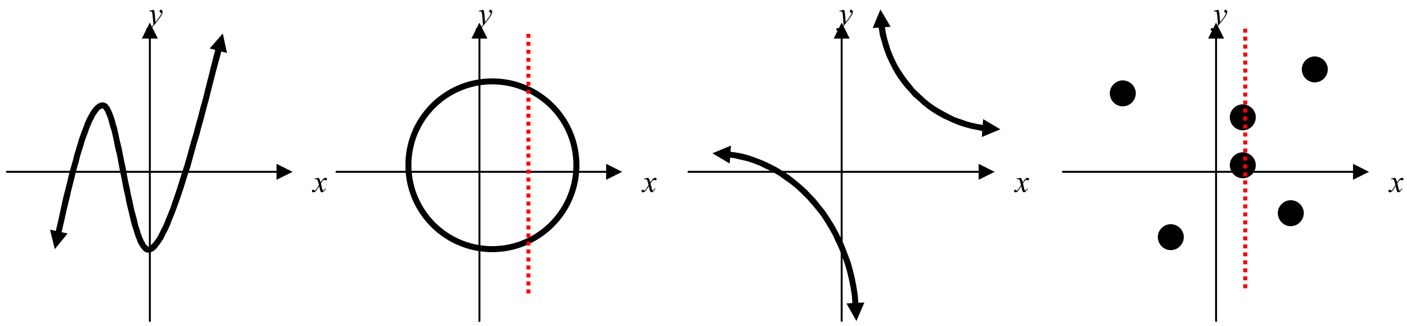


Function: - a special relation that must satisfy the following ***two conditions***:

a. The graph is ***continuous*** (no break unless already stated).

b. For ***each input***, there is ***only one unique output*** – known as "***one-to-one***" or ***Vertical Line Test***.
(*Vertical Line Test* – If a vertical line moves from left to right of the graph and it did not cross the graph at two different points, then we can say the graph passed the vertical line test or "one-to-one").

Example 3: State whether each of the graphs below is a function. Provide reasons.



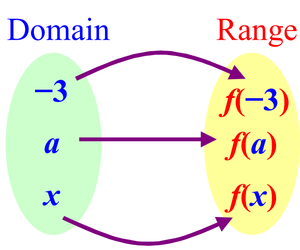
Function: Continuous and pass Vertical Line

NOT a Function: Does NOT pass Vertical Line Test

Function (at each branch): Continuous and pass Vertical Line Test

NOT a Function: Does NOT pass Vertical Line Test

Function Notation: - a way to express an equation to denote that it is a function (satisfies requirements of continuity and vertical line test).
- **instead of writing y, we can write $f(x)$** and we can say **"Function f of x"** or **"f as a function of x"**. Other letters and variables can be used, such as $g(t)$ or $h(r)$.
- a number can be put in to replace x in $f(x)$ for the purpose of substitution.



Arrow diagram of function f

$y = f(x)$

$f(x)$ is NOT $f \times x$ or f multiplies x

Machine diagram of function f

Domain: - all possible numbers in a set that is allowed to input into function f.
- in many ways, it is similar to list the ***restrictions*** of an expression.

Examples: $f(x) = \dfrac{1}{x}$ has a domain of $x \neq 0$ (cannot divide by 0).

$f(x) = \sqrt{x}$ has a domain of $x \geq 0$ (cannot take a square root of a negative number).

Range: - all possible numbers in a set that would be the output of function f.

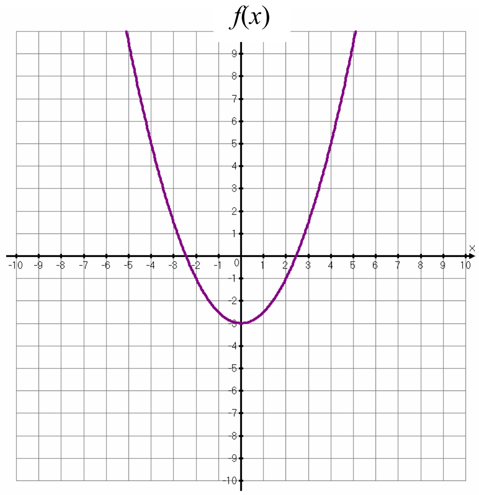
Examples: $f(x) = \dfrac{1}{x}$ has a range of $f(x) \neq 0$ (1 divided by anything cannot result in a zero).

$f(x) = \sqrt{x}$ has a domain of $f(x) \geq 0$ (there are no negative results from a square root operation).

Example 4: Given $f(x) = \frac{1}{2}x^2 - 3$, set up a table of values for $x = -3$ to $x = 3$. Graph $f(x)$ and state its domain and range.

a. Evaluate $f(8)$. b. Find x when $f(x) = 159$. c. Evaluate $f(a)$. d. Evaluate $f(2n - 1)$.

x	$f(x)$
−3	$\frac{1}{2}(-3)^2 - 3 = \frac{3}{2}$
−2	$\frac{1}{2}(-2)^2 - 3 = -1$
−1	$\frac{1}{2}(-1)^2 - 3 = -\frac{5}{2}$
0	$\frac{1}{2}(0)^2 - 3 = -3$
1	$\frac{1}{2}(1)^2 - 3 = -\frac{5}{2}$
2	$\frac{1}{2}(2)^2 - 3 = -1$
3	$\frac{1}{2}(3)^2 - 3 = \frac{3}{2}$



From the graph, we can see that there is ***no restriction on x*** (x values can be all positive and negative numbers). Hence, **Domain is $x \in R$**.

There is a ***minimum value*** where y cannot be below −3. Therefore, the **Range is $f(x) \geq -3$**.

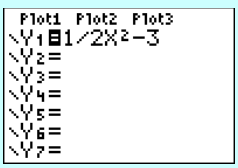
a. **Algebraically**:

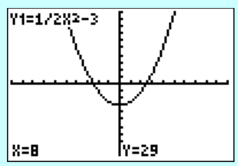
$f(x) = \frac{1}{2}x^2 - 3$

$f(8) = \frac{1}{2}(8)^2 - 3$

$\boxed{f(8) = 29}$

Graphically: Y= GRAPH TRACE

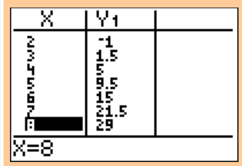




Type 8

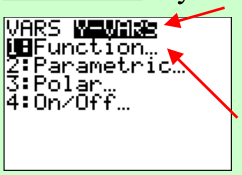
ENTER

Table of Values: 2nd TABLE GRAPH



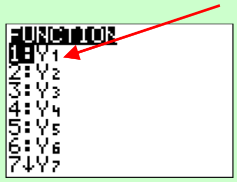
Functions: (after entering function in Y_1 of the Y= screen)

VARS Select **Y-VARS** by using ▶

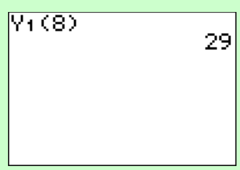


Select Option **1: Function**

Select Option **1: Y_1**



Type in **(8)** and press ENTER



b. $f(x) = \frac{1}{2}x^2 - 3$ $f(x) = 159$

$159 = \frac{1}{2}x^2 - 3$

$159 + 3 = \frac{1}{2}x^2$

$2(162) = x^2$

$\pm\sqrt{324} = x$

$\boxed{x = \pm 18}$

c. $f(x) = \frac{1}{2}x^2 - 3$

$f(a) = \frac{1}{2}(a)^2 - 3$

$\boxed{f(a) = \frac{1}{2}a^2 - 3}$

d. $f(x) = \frac{1}{2}x^2 - 3$

$f(2n - 1) = \frac{1}{2}(2n - 1)^2 - 3$

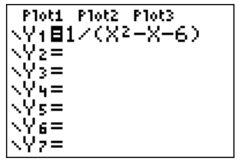
$f(2n - 1) = \frac{1}{2}(4n^2 - 4n + 1) - 3$

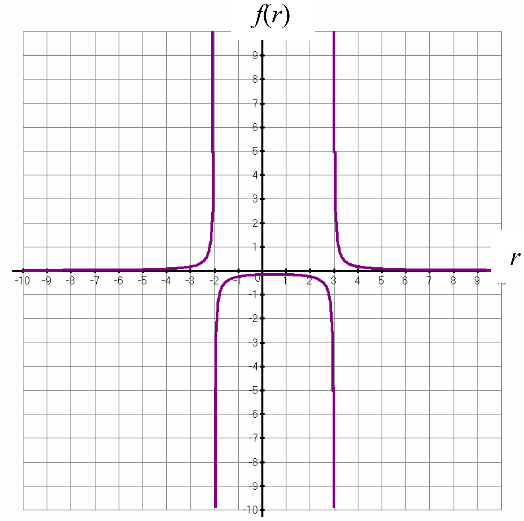
$f(2n - 1) = 2n^2 - 2n + \frac{1}{2} - 3$

$\boxed{f(2n - 1) = 2n^2 - 2n - \frac{5}{2}}$

Example 5: Find the domain of the functions below. Graph to verify.

a. $f(r) = \dfrac{1}{r^2 - r - 6}$





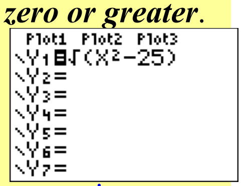
The denominator cannot be zero.

$$r^2 - r - 6 \neq 0$$
$$(r - 3)(r + 2) \neq 0$$
$$r - 3 \neq 0 \quad \text{and} \quad r + 2 \neq 0$$

Domain: $\{\, r \mid r \neq 3, r \neq -2\}$

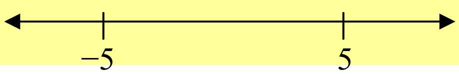
b. $g(t) = \sqrt{t^2 - 25}$

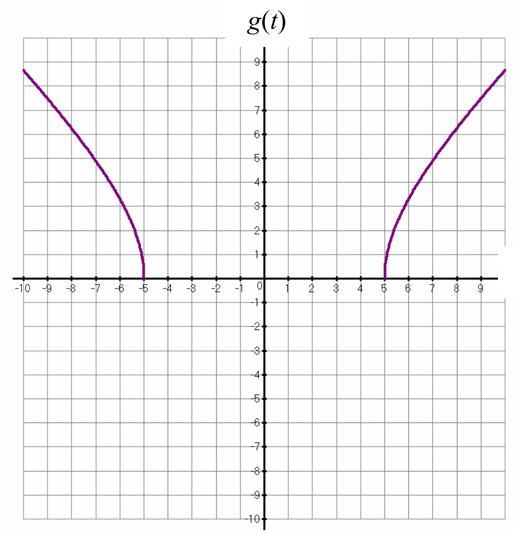
The radicand of a square root must be zero or greater.



$$t^2 - 25 \geq 0$$
$$(t - 5)(t + 5) \geq 0$$
$t = 5$ and $t = -5$ (boundary points)

Using a number line and a table to evaluate regions.

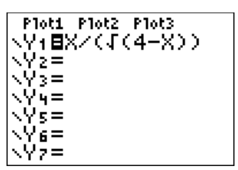


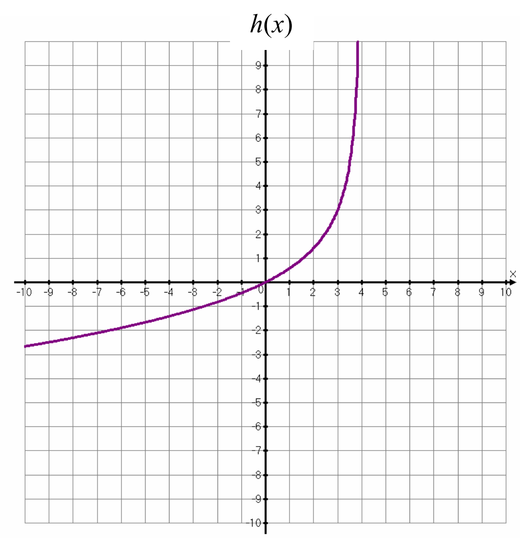


	$(t-5)$	$(t+5)$	$(t-5)(t+5)$
$t \leq -5$	−	−	+
$-5 \leq t \leq 5$	−	+	−
$t \geq 5$	+	+	+

Domain: $\{\, t \mid t \leq -5$ or $t \geq 5\}$ or $(-\infty, -5] \cup [5, \infty)$

c. $h(x) = \dfrac{x}{\sqrt{4 - x}}$





The denominator cannot be zero **and its *radicand must be zero or greater.*** → ***radicand in the denominator must be greater than zero.***

$$4 - x > 0$$
$$-x > -4$$
$$x < 4$$

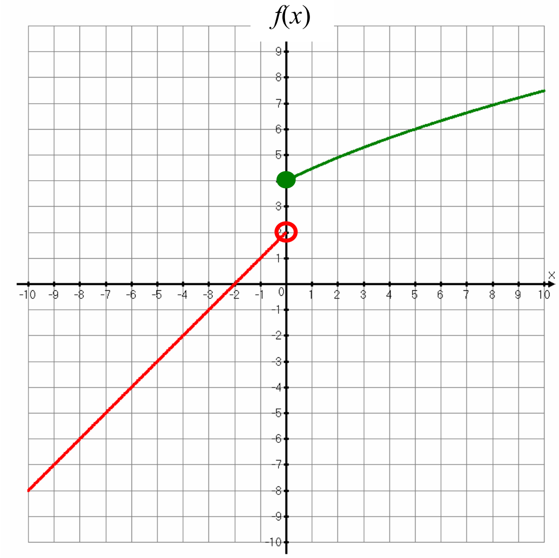
Domain: $\{\, x \mid x < 4\}$ or $(-\infty, 4)$

Piecewise Function: - a function that consists of two or more "sub-functions" which are applied at different domains.

Example: $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 2\sqrt{x+4} & \text{if } x \geq 0 \end{cases}$

When $x < 0$, we follow the first "sub-function", which is $x + 2$.

When $x \geq 0$, we follow the first "sub-function", which is $2\sqrt{x+4}$.



Example 6: For the function, $f(x) = \begin{cases} 1-2x & \text{if } x \leq 1 \\ x^2 - 3 & \text{if } x > 1 \end{cases}$, evaluate $f(-3)$, $f(1)$ and $f(4)$.

When $x = -3$, it is within the condition $x \leq 1$, hence we follow

$f(x) = 1 - 2x$
$f(-3) = 1 - 2(-3)$
$f(-3) = 7$

When $x = 1$, it is within the condition $x \leq 1$, hence we follow

$f(x) = 1 - 2x$
$f(1) = 1 - 2(1)$
$f(1) = -1$

When $x = 4$, it is within the condition $x > 1$, hence we follow

$f(x) = x^2 - 3$
$f(4) = (4)^2 - 3$
$f(4) = 13$

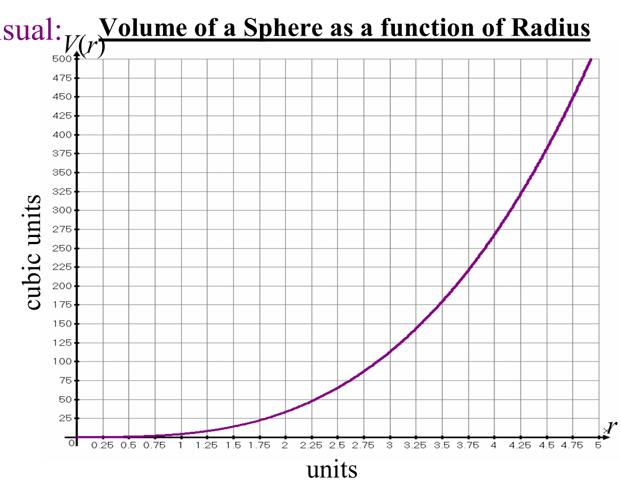
Four Ways to Represent a Function:
Verbal (Words), Algebraic (Function Notation), Visual (Graph), and Numerical (Table of Values)

Example 7: The volume of a sphere can be calculated by $\frac{4}{3}\pi r^3$, where r is its radius. Represent this function in four different ways.

Verbal: ***V*(*r*) is the volume in terms of radius *r*.**

Algebraic: $V(r) = \frac{4}{3}\pi r^3$

Visual:



Numerical:

r	$V(r)$
0 unit	0 unit^3
1 unit	$\frac{4}{3}\pi$ unit^3
2 units	$\frac{32}{3}\pi$ unit^3
3 units	36π unit^3

3-1 Assignment: pg. 215–218 #3, 7, 11, 17, 21, 23, 37, 41, 45, 47, 55, 61, 62a & b, 67; Honours: #27, 29, 33, 51, 69, 71

3-2 Graphs of Functions (Part 1)

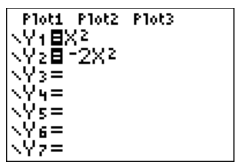
Graphing Functions:

1. Note the **Restrictions to *Domain* and *Range***.
2. Make a ***Table of Values***. **OR**
3. ***Recognize the Type of Functions*** (powers like quadratic and cubic, root, absolute value, reciprical, greatest integer, exponential, logarithmic, or trigonometric) and do the necessary ***Transformation(s)***.
4. For **Piecewise Function**, treat it as a **combination of multiple "Sub-functions" or Pieces**. Remember to **evaluate the Boundary Point(s)** of each piece.
5. **Verify** by using a **Graphing Calculator** if available.

Example 1: Sketch the graphs of the following functions by making tables of values. State their domains and ranges.

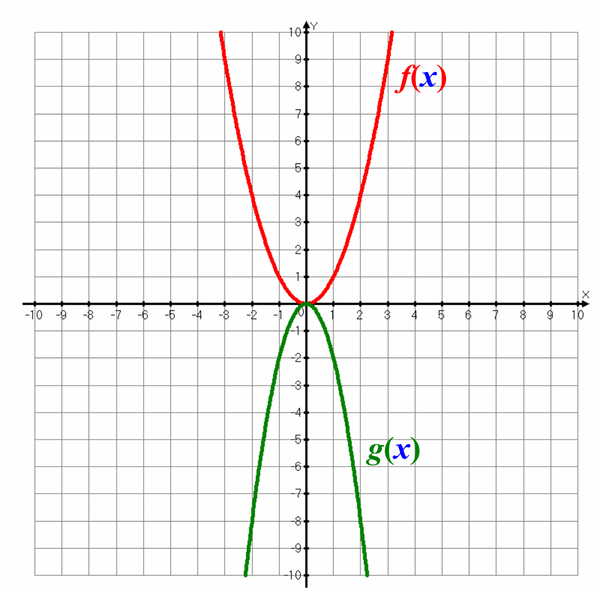
a. $f(x) = x^2$ and $g(x) = -2x^2$

x	$f(x)$	$g(x)$
−3	**9**	**−18**
−2	**4**	**−8**
−1	**1**	**−2**
0	**0**	**0**
1	**1**	**−2**
2	**4**	**−8**
3	**9**	**−18**



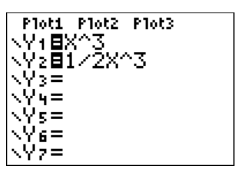
For $f(x)$:
Domain: $x \in R$
Range: $f(x) \geq 0$

For $g(x)$:
Domain: $x \in R$
Range: $g(x) \leq 0$



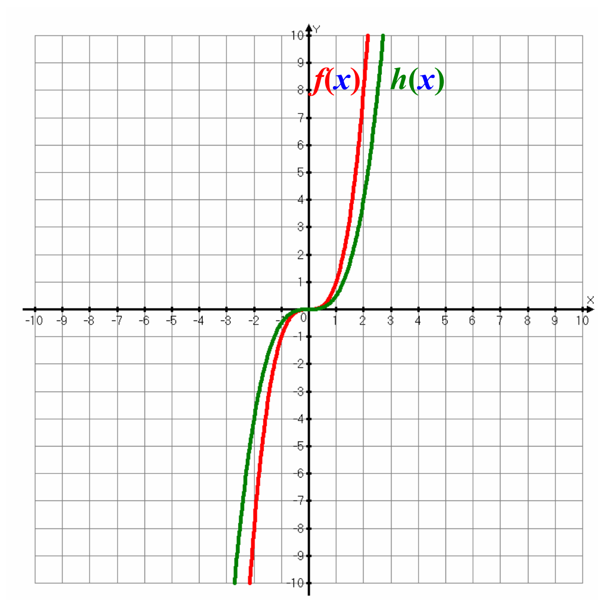
b. $f(x) = x^3$ and $h(x) = \frac{1}{2}x^3$

x	$f(x)$	$h(x)$
−3	**−27**	**−13.5**
−2	**−8**	**−4**
−1	**−1**	**−½**
0	**0**	**0**
1	**1**	**½**
2	**8**	**4**
3	**27**	**13.5**



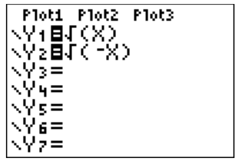
For $f(x)$:
Domain: $x \in R$
Range: $f(x) \in R$

For $h(x)$:
Domain: $x \in R$
Range: $h(x) \in R$



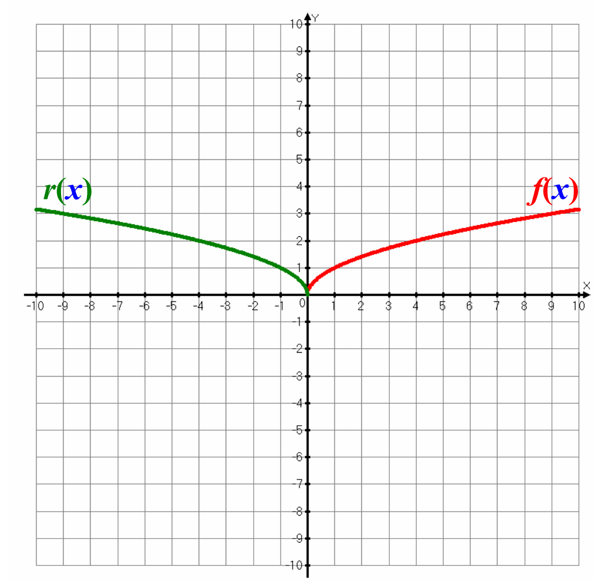
c. $f(x) = \sqrt{x}$ and $r(x) = \sqrt{-x}$

x	$f(x)$	$r(x)$
−3	∅	$\sqrt{3} \approx 1.732$
−2	∅	$\sqrt{2} \approx 1.414$
−1	∅	1
0	0	0
1	1	∅
2	$\sqrt{2} \approx 1.414$	∅
3	$\sqrt{3} \approx 1.732$	∅



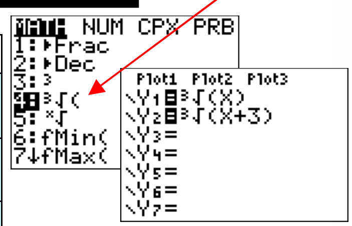
For $f(x)$:
Domain: $x \geq 0$
Range: $f(x) \geq 0$

For $r(x)$:
Domain: $x \leq 0$
Range: $r(x) \geq 0$



d. $f(x) = \sqrt[3]{x}$ and $p(x) = \sqrt[3]{x+3}$

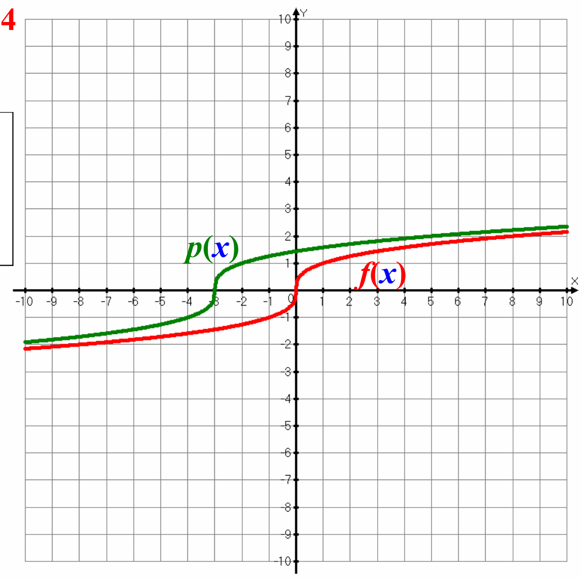
MATH Select Option **4**



x	$f(x)$	$p(x)$
−3	$\sqrt[3]{-3} \approx -1.442$	0
−2	$\sqrt[3]{-2} \approx -1.259$	1
−1	−1	$\sqrt[3]{2} \approx 1.259$
0	0	$\sqrt[3]{3} \approx 1.442$
1	1	$\sqrt[3]{4} \approx 1.587$
2	$\sqrt[3]{2} \approx 1.259$	$\sqrt[3]{5} \approx 1.710$
3	$\sqrt[3]{3} \approx 1.442$	$\sqrt[3]{6} \approx 1.817$

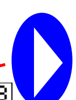
For $f(x)$:
Domain: $x \in R$
Range: $f(x) \in R$

For $p(x)$:
Domain: $x \in R$
Range: $p(x) \in R$

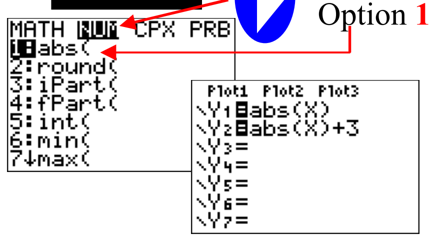


e. $f(x) = |x|$ and $g(x) = |x| + 3$

MATH



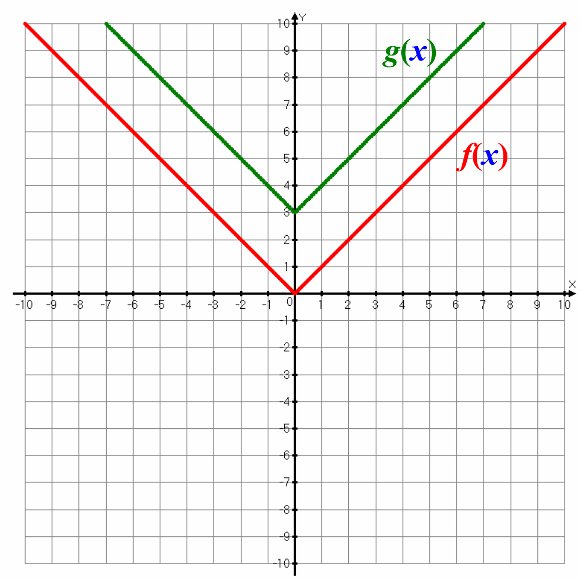
Select Option **1**



x	$f(x)$	$g(x)$
−3	3	6
−2	2	5
−1	1	4
0	0	3
1	1	4
2	2	5
3	3	6

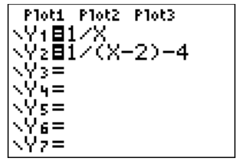
For $f(x)$:
Domain: $x \in R$
Range: $f(x) \geq 0$

For $g(x)$:
Domain: $x \in R$
Range: $g(x) \geq 3$



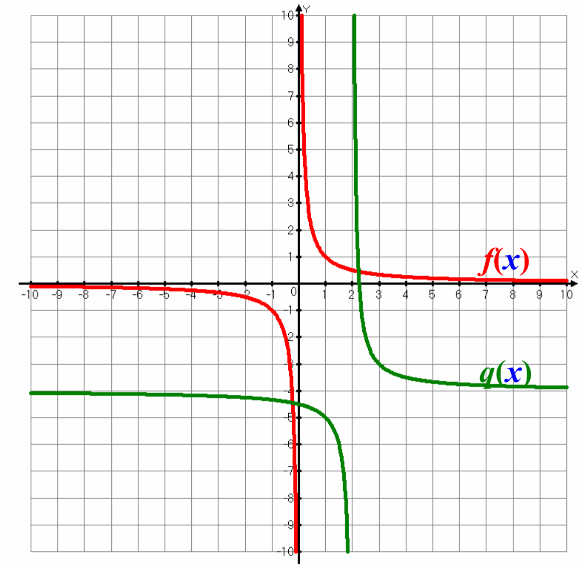
f. $f(x) = \frac{1}{x}$ and $q(x) = \frac{1}{x-2} - 4$

x	$f(x)$	$q(x)$
−3	−⅓	−4.2
−2	−½	−4.25
−1	−1	−4.333
0	undefined	−4.5
1	1	−5
2	½	undefined
3	⅓	−3



For $f(x)$:
Domain: $x \neq 0$
Range: $f(x) \neq 0$

For $q(x)$:
Domain: $x \neq 2$
Range: $q(x) \neq -4$



Graphing Piecewise Functions:

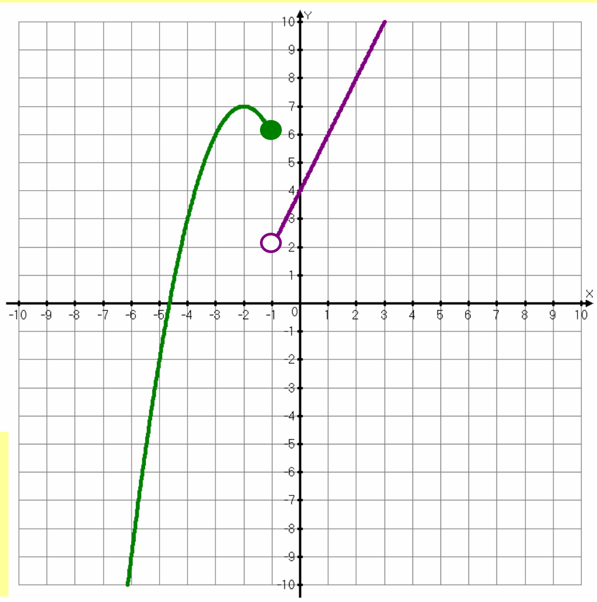
1. **Identify** the **Boundary** of each "***Sub-function***" or ***Piece***, and **find their outputs**. Be sure to ***pay attention to the inequality symbol of each boundary***.
2. **Make a table of values** of a few points beyond each boundary and graph the function.

Example 2: Make a table of values and graph the function, $f(x) = \begin{cases} 7-(x+2)^2 & \text{if } x \le -1 \\ 2x+4 & \text{if } x > -1 \end{cases}$

Since the boundary is at $x = -1$, we select three points before and after $x = -1$ for our table of values.

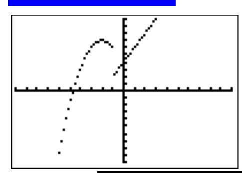
x	$f(x)$
−4	$7-((-4)+2)^2 = 3$
−3	$7-((-3)+2)^2 = 6$
−2	$7-((-2)+2)^2 = 7$
−1	$7-((-1)+2)^2 = 6$
0	$2(0)+4 = 4$
1	$2(1)+4 = 6$
2	$2(2)+4 = 8$

Note that since $x \le -1$ is on the sub-function, $7-(x+2)^2$, it is that piece that gets the **closed circle**.

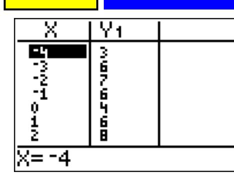


After entering the function, (see below)

GRAPH



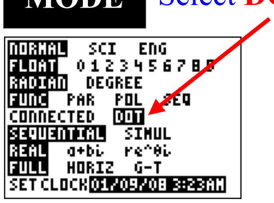
2nd **TABLE** **GRAPH**

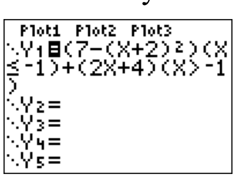


To Graph Piecewise Function with a Graphing Calculator.

MODE Select **DOT**

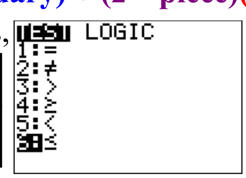
Type in Function. Use brackets around each "sub-function" follow by the boundary in bracket. Between each piece, use an addition sign.





(1st piece)(1st boundary) + (2nd piece)(2nd boundary)

For inequality symbols, **2nd** **TEST** **MATH**



Greatest Integer Function: - the output is always the previous integer if the input is a decimal number.
- if the input is an integer, the same integer would be the output.
- sometimes refer to as the "***floor function***" or "***step function***"

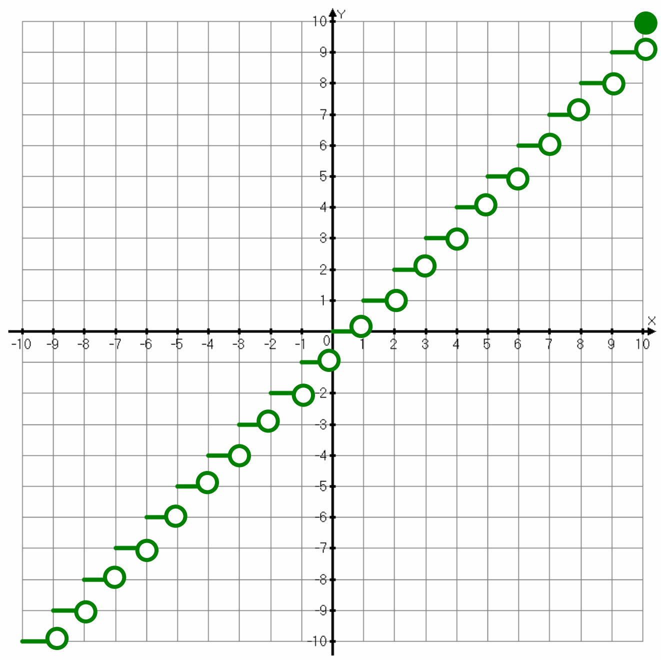
$f(x) = [x]$ or $\lfloor x \rfloor$ or $[[x]]$ or $\text{int}(x)$

Examples: [[3]] = 3 ; int(−4) = −4 ; int(−1.321) = −2 ; [[5.782]] = 5 ; int(0.527) = 0 ; [[−0.004]] = −1

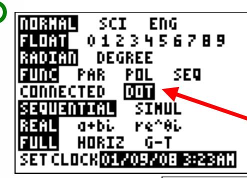
Graphing Step Function:

Example 3: Graph the step function, $f(x) = [[x]]$ by making a table of values first and then by using a graphing calculator.

x	$f(x)$
−2.5	[[−2.5]] = **−3**
−2	[[−2]] = **−2**
−1.5	[[−1.5]] = **−2**
−1	[[−1]] = **−1**
−0.5	[[−0.5]] = **−1**
0	[[0]] = **0**
0.5	[[0.5]] = **0**
1	[[1]] = **1**
1.5	[[1.5]] = **1**
2	[[2]] = **2**
2.5	[[2.5]] = **2**



To Graph Step Function with a Graphing Calculator:

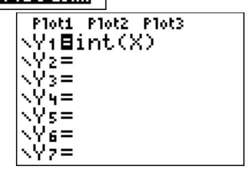


MODE

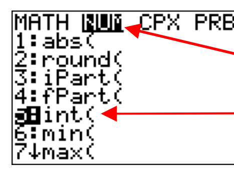
Select **DOT**

Enter function

Y=

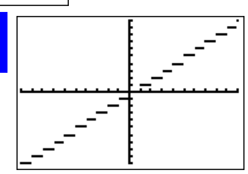


For int function, **MATH**



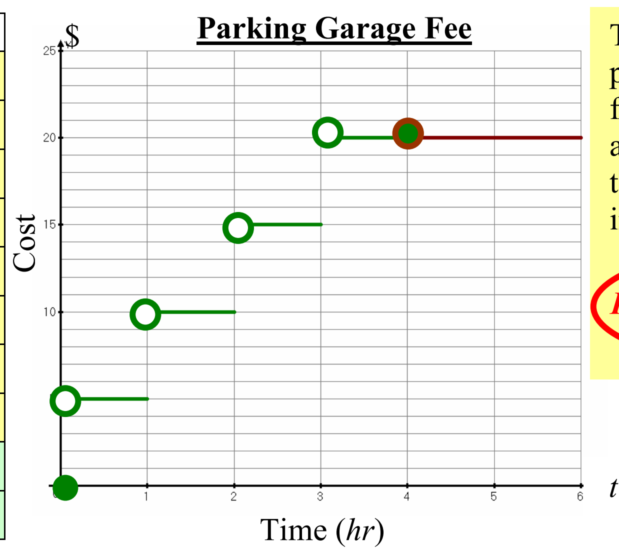
Select Option **5**

GRAPH



Example 4: A parking garage charges $5 per hour or less than the full hour up to a daily maximum of four hours. Graph this fee structure and express it in a function notation.

t in hrs	$F(t)$ in $
0	**0** = −5[[−(0)]]
0.5	**5** = −5[[−(0.5)]]
1	**5** = −5[[−(1)]]
1.5	**10** = −5[[−(1.5)]]
2	**10** = −5[[−(2)]]
2.5	**15** = −5[[−(2.5)]]
3	**15** = −5[[−(3)]]
4	**20** = −5[[−(4)]]
4.5	**20**
5	**20**



This relation will be a piecewise as well as a step function. (Function is different at various conditions and needs to round decimal to the next integer.)

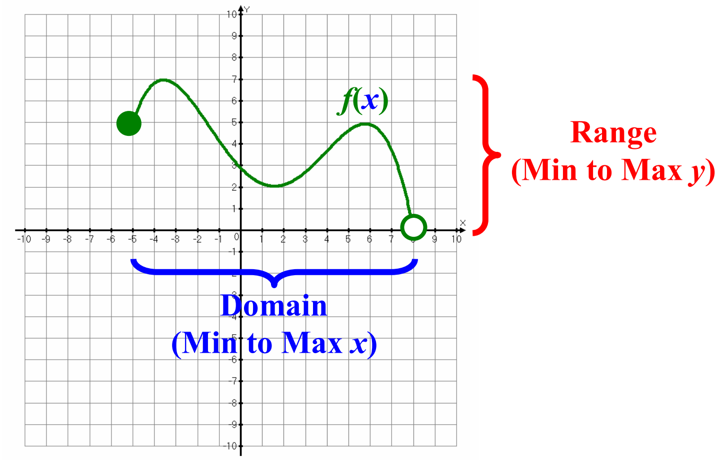
$$F(t) = \begin{cases} -5[[-t]] & \text{if } 0 \le t \le 4 \\ 20 & \text{if } t > 4 \end{cases}$$

3-2 (Part 1) Assignment: pg. 227−228 #5, 11, 19, 31, 33, 41, 45, 53; Honours: #17, 49, 51

3-2 Graphs of Functions (Part 2)

On a graph, Coordinates $(x, y) = (x, f(x))$

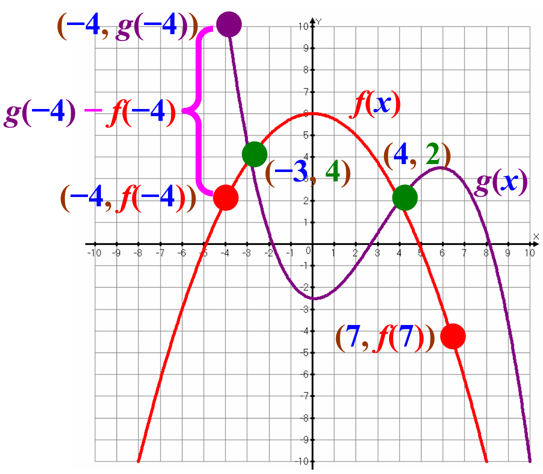
Example 1: Given the graph below, state its domain and range.



Domain: $-5 \le x < 8$

Range: $0 < f(x) \le 7$

Example 2: Given the graphs of the functions, $f(x)$ and $g(x)$ below, find the following.



a. $f(7)$

When $x = 7$, $f(7) = -6$

Note: $(7, f(7)) = (7, -6)$

b. $g(-4) - f(-4)$

When $x = -4$, $g(-4) = 10$
When $x = -4$, $f(-4) = 2$

$g(-4) - f(-4) = 10 - 2 =$ **8**

c. all x when $f(x) = g(x)$

$f(x) = g(x)$ happens when the <u>two graphs intersect</u> (sharing the same y).

$f(x) = g(x)$ when

$x = -3$ and $x = 4$

d. all x when $f(x) > g(x)$

$f(x) > g(x)$ happens when $f(x)$ is <u>higher</u> than $g(x)$.

$f(x) > g(x)$ when

$-3 < x < 4$

<u>Equations that are not Functions</u>:

- When the output, **<u>y, has an *Even Exponent*</u>** or **<u>y, is in an *Absolute Value* bracket</u>**.
- Because in both cases, **<u>Solving for y will Generate TWO Solutions</u> ($\pm\sqrt{y}$ or $\pm y$)** for any one particular x.

Example 3: Determine if the following equations are functions algebraically. Graph the results to verify.

a. $3\,|y| = 6x - 15$

$$|y| = \frac{6x-15}{3}$$

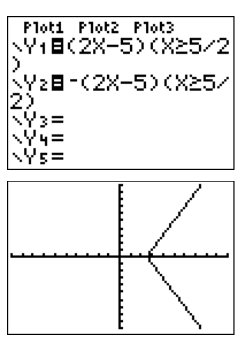
$$|y| = 2x - 5$$

$$\pm y = 2x - 5$$

Domain: $2x - 5 \ge 0$ **(because $|y| \ge 0$)**

Domain: $x \ge \frac{5}{2}$

Not a function (fails the Vertical Line Test)

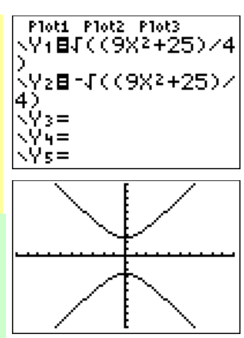


b. $4y^2 = 9x^2 + 25$

$$y^2 = \frac{9x^2+25}{4}$$

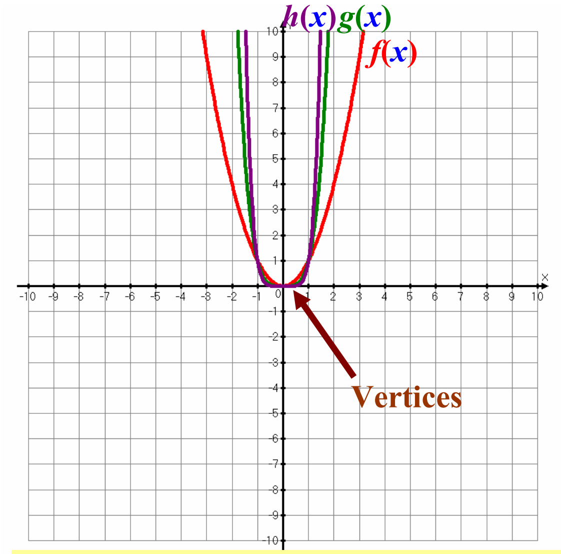
$$y = \pm\sqrt{\frac{9x^2+25}{4}}$$

Not a function (fails the Vertical Line Test)



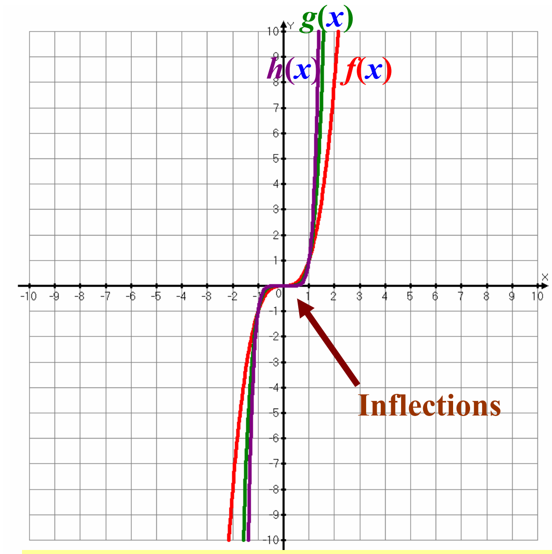
Example 4: Graph the following functions and compare them.

a. $f(x) = x^2$; $g(x) = x^4$; $h(x) = x^6$



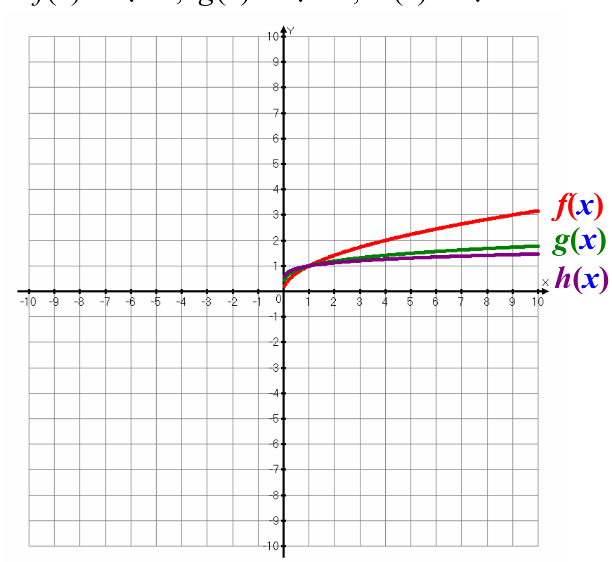
All **Even Power Functions** **(x^2, x^4, x^6, …)** are in the shape of a ***Vertical Parabola***. ***The higher the power, the more vertically stretched is the graph, and it is flatter near the Vertex*** (parabola's highest or lowest point).

b. $f(x) = x^3$; $g(x) = x^5$; $h(x) = x^7$



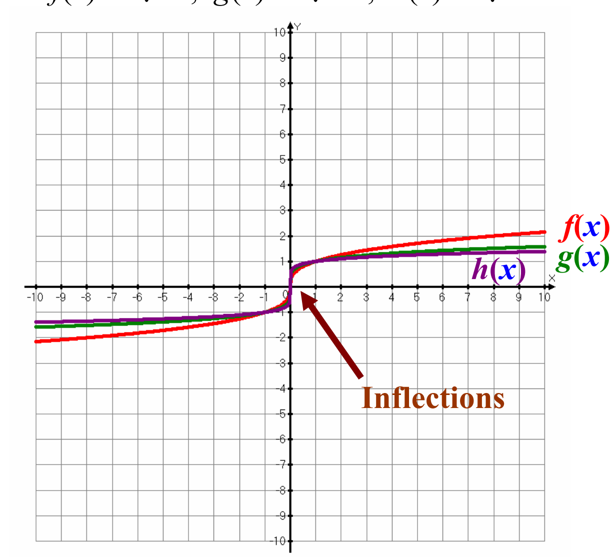
All **Odd Power Functions** **(x^3, x^5, x^7, …)** have an ***Inflection***. ***The higher the power, the more vertically stretched is the graph, and it is flatter near the Inflection***.

c. $f(x) = \sqrt{x}$; $g(x) = \sqrt[4]{x}$; $h(x) = \sqrt[6]{x}$



All **Even Index Root Functions** **($\sqrt{x}, \sqrt[4]{x}, \sqrt[6]{x}, \ldots$)** are in the shape of a ***Half Horizontal Parabola***. ***The higher the index, the more horizontally stretched is the graph.***

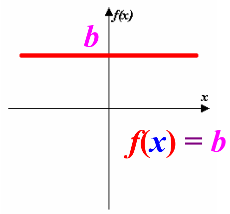
d. $f(x) = \sqrt[3]{x}$; $g(x) = \sqrt[5]{x}$; $h(x) = \sqrt[7]{x}$



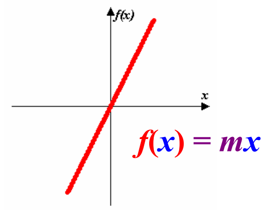
All **Odd Index Root Functions** **($\sqrt[3]{x}, \sqrt[5]{x}, \sqrt[7]{x}, \ldots$)** have an ***Inflection***. ***The higher the power, the more horizontally stretched is the graph, and it is flatter near the Inflection***.

Summary of Types of Functions: (see page 226 of textbook)

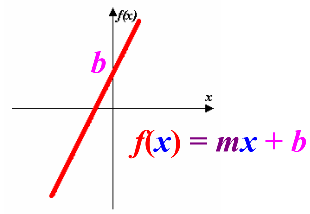
Linear Functions $f(x) = mx + b$



Domain: $x \in R$
Range: $f(x) \in R$

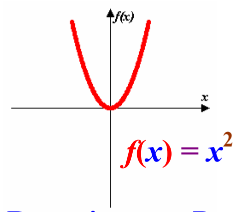


Domain: $x \in R$
Range: $f(x) \in R$

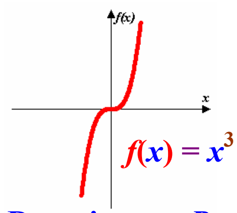


Domain: $x \in R$
Range: $f(x) \in R$

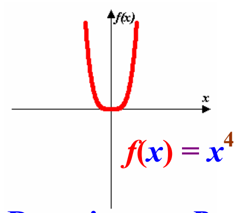
Power Functions $f(x) = x^n$ where $n > 1$ and $n \in N$



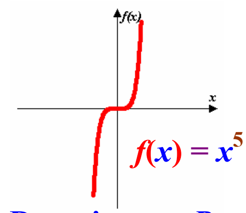
Domain: $x \in R$
Range: $f(x) \geq 0$



Domain: $x \in R$
Range: $f(x) \in R$

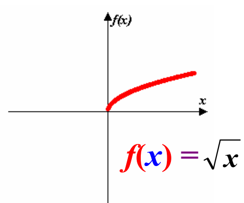


Domain: $x \in R$
Range: $f(x) \geq 0$

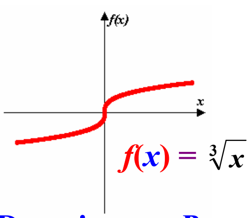


Domain: $x \in R$
Range: $f(x) \in R$

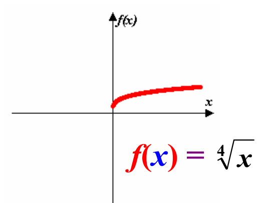
Root Functions $f(x) = \sqrt[n]{x}$ where $n \geq 2$ and $n \in N$



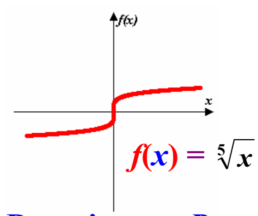
Domain: $x \geq 0$
Range: $f(x) \geq 0$



Domain: $x \in R$
Range: $f(x) \in R$

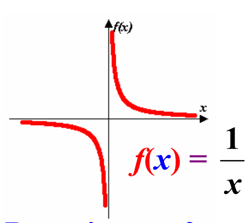


Domain: $x \geq 0$
Range: $f(x) \geq 0$

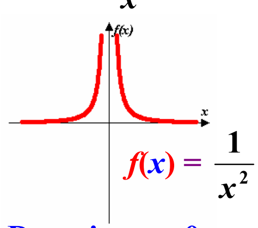


Domain: $x \in R$
Range: $f(x) \in R$

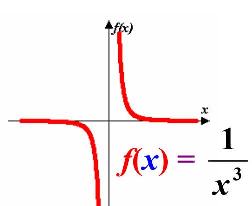
Reciprocal Functions $f(x) = \dfrac{1}{x^n}$ where $n \in N$



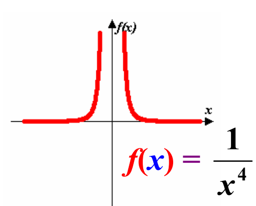
Domain: $x \neq 0$
Range: $f(x) \neq 0$



Domain: $x \neq 0$
Range: $f(x) > 0$

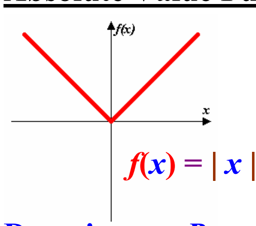


Domain: $x \neq 0$
Range: $f(x) \neq 0$



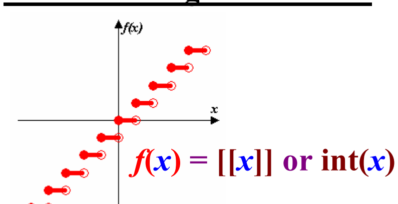
Domain: $x \neq 0$
Range: $f(x) > 0$

Absolute Value Functions



Domain: $x \in R$
Range: $f(x) \geq 0$

Greatest Integer Functions



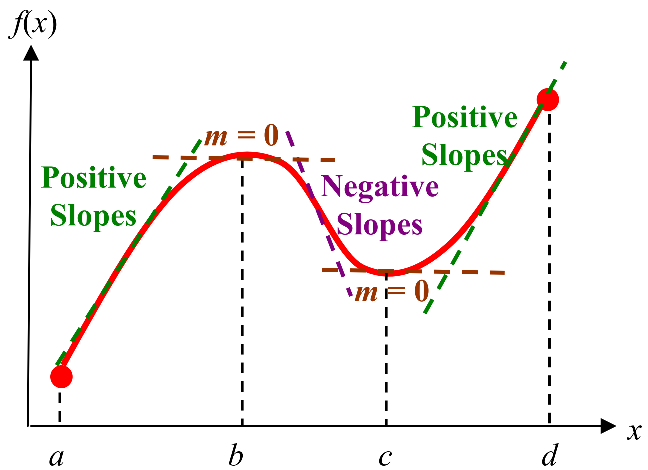
Domain: $x \in R$
Range: $f(x) \in I$

3-2 (Part 2) Assignment:
pg. 228−230 #23, 25, 55, 57, 59, 61, 63, 67, 70, 83, 85;
Honours: #81, 82, 94

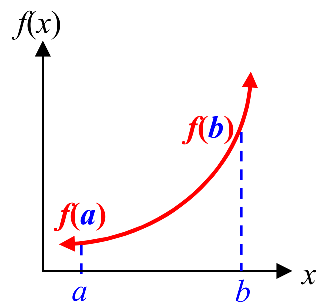
3-3 Increasing and Decreasing Functions; Average Rate of Change

Increasing Function: - the interval of the function where the ***Slope*** *over the interval is* ***Positive***.

Decreasing Function: - the interval of the function where the ***Slope*** *over the interval is* ***Negative***.



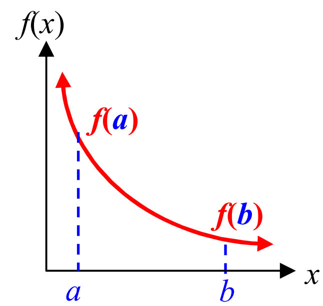
Positive Slopes ($m > 0$) when x is [a, b) and (c, d]
Negative Slopes ($m < 0$) at $b < x < c$
Zero Slopes at $x = b$ and $x = c$



Increasing Functions

When $f(a) < f(b)$ if $a < b$

Slope: $\dfrac{f(b) - f(a)}{b - a} > 0$

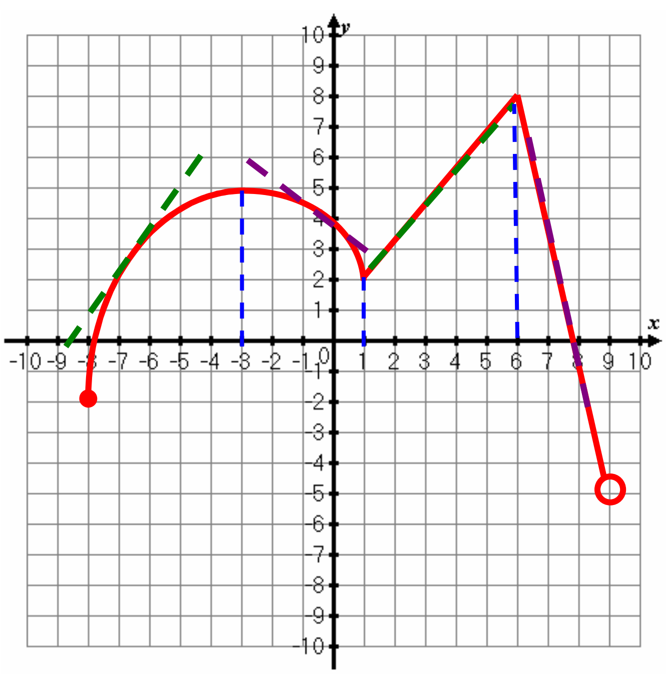


Decreasing Functions

When $f(a) > f(b)$ if $a < b$

Slope: $\dfrac{f(b) - f(a)}{b - a} < 0$

Example 1: Given the graph of a function below, determine the intervals on which the function is



a. increasing.

Positive Slopes at

[−8, −3) and (1, 6)

$-8 \leq x < -3$ and $1 < x < 6$

b. decreasing.

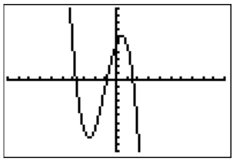
Negative Slopes at

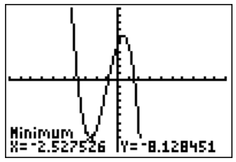
(−3, 1) and (6, 9)

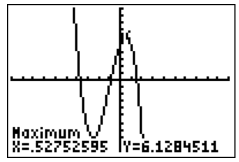
$-3 < x < 1$ and $6 < x < 9$

Example 2: Using a graphing calculator, graph $f(x) = -x^3 - 3x^2 + 4x + 5$ To the nearest hundredth, state the intervals on which the function is increasing, and on which the function is decreasing.

GRAPH | **Find Local Minimum (run minimum)** | **Find Local Maximum (run maximum)**



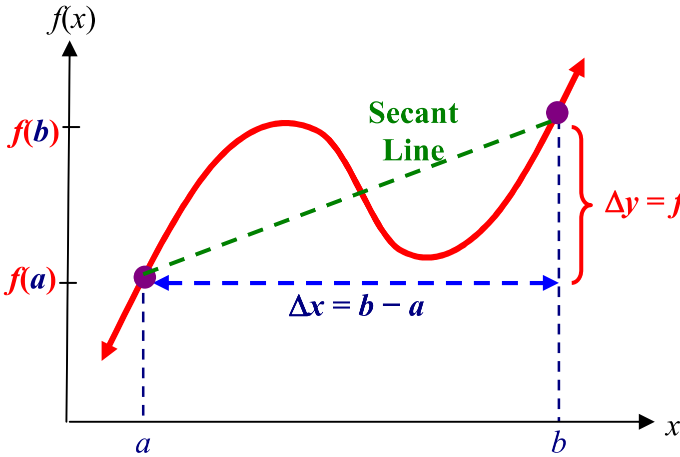




From the local min and max, we can see that $f(x)$ is

increasing at $-2.53 < x < 0.53$
decreasing at $x < -2.53$ and $x > 0.53$

Average Rate of Change: - the slope of the **secant line** between two given inputs of the a function.

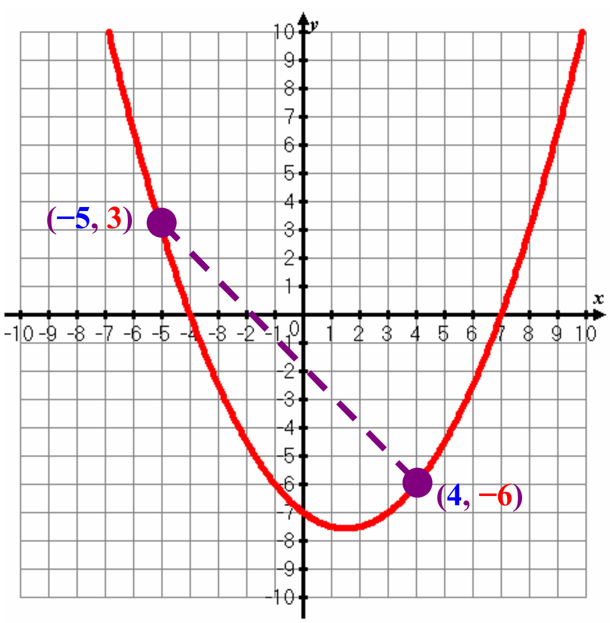


Average Rate of Change = $m = \dfrac{\Delta y}{\Delta x}$

Average Rate of Change = $\dfrac{f(b) - f(a)}{b - a}$

It is the slope of the secant line between the points $(a, f(a))$ and $(b, f(b))$

Example 3: From the graph of a function below, find the average rate of change between $x = -5$ and $x = 4$.



Average Rate of Change = $\dfrac{f(b) - f(a)}{b - a}$

between the points (−5, 3) and (4, −6).

Average Rate of Change = $\dfrac{(-6)-(3)}{(4)-(-5)} = \dfrac{-9}{9}$

Average Rate of Change = −1

Example 4: Find average rate of change for $f(x) = 8 - x^2$ between,

a. $x = -4$ and $x = 1$

$f(-4) = 8 - (-4)^2$ $f(-4) = -8$
$f(1) = 8 - (1)^2$ $f(1) = 7$

$$\text{Avg Rate of Change} = \frac{f(b) - f(a)}{b - a}$$

$$\text{Avg Rate of Change} = \frac{f(1) - f(-4)}{(1) - (-4)}$$

$$\text{Avg Rate of Change} = \frac{7 - (-8)}{(1) - (-4)} = \frac{15}{5}$$

Average Rate of Change = 3

b. $x = a$ and $x = a - h$

$f(a) = 8 - (a)^2$ $f(a) = (8 - a^2)$

$f(a - h) = 8 - (a - h)^2$
$f(a - h) = 8 - (a^2 - 2ah + h^2)$
$f(a - h) = (8 - a^2 + 2ah - h^2)$

$$\text{Avg Rate of Change} = \frac{f(b) - f(a)}{b - a} = \frac{f(a-h) - f(a)}{(a-h) - (a)}$$

$$\text{Avg Rate of Change} = \frac{(8 - a^2 + 2ah - h^2) - (8 - a^2)}{(a-h) - (a)}$$

$$\text{Avg Rate of Change} = \frac{8 - a^2 + 2ah - h^2 - 8 + a^2}{h}$$

$$\text{Avg Rate of Change} = \frac{2ah - h^2}{h} = \frac{h(2a - h)}{h}$$

Average Rate of Change = $2a - h$

Example 5: The table shows the revenue of an electronic manufacturing company for a period of ten years.

a. Find average rate of change of revenue between 2000 and 2005.
b. For what period(s) of time is the revenue increasing? For what period(s) of time is it decreasing?

Year	Revenue (in $ Millions)
1998	7.23
1999	9.56
2000	12.22
2001	10.86
2002	8.74
2003	15.24
2004	23.18
2005	38.25
2006	51.94
2007	65.35

a. Between 2000 and 2005, revenues are at $12.22 million and $38.25 million.

(2000, $12.22 million) and (2000, $12.22 million)

$$\text{Avg Rate of Change} = \frac{\$38.25 \text{ million} - \$12.22 \text{ million}}{2005 - 2000}$$

Avg Rate of Change = $5.206 million / year (for 2000-2005)

b. From the table, we can see that revenues increased between 1998 to 2000 and between 2002 to 2007. However, revenues decreased between 2000 to 2002. Hence,

R increased when $1998 \le t \le 2000$ and $2002 \le t \le 2007$
R decreased when $2000 \le t \le 2002$

3-3 Assignment: pg. 239−241 #3, 11 (exactly with calc), 15, 17, 21, 25, 31, 33;
Honours: #27, 38, 39

3-4 Transformations of Functions (Part 1)

Transformation: - a modification of a function or a relation.

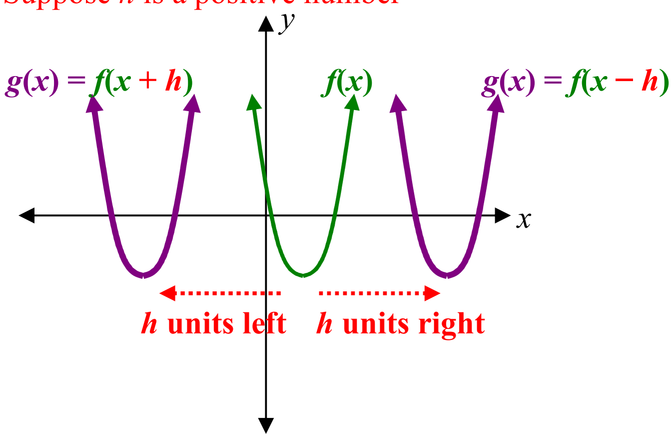
Translation: - the transformation of a function or relation that involves simple horizontal and/or vertical "slide" of the graph. The **shape and size** of the original graph is **not altered**.

$$g(x) = f(x + h) + k$$

h **= amount of horizontal movement**	***h*** **> 0 (move left);** ***h*** **< 0 (move right)**
k **= amount of vertical movement**	***k*** **> 0 (move up);** ***k*** **< 0 (move down)**

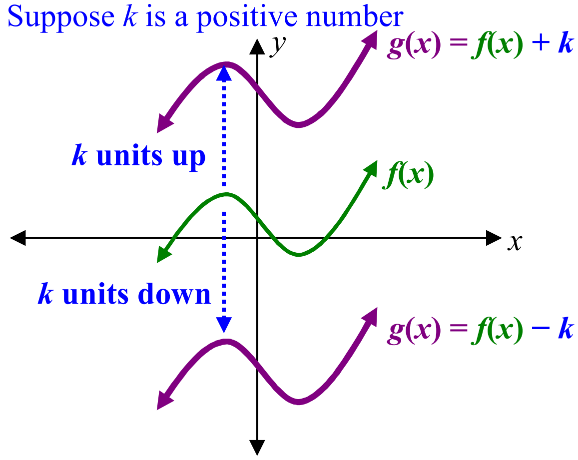
Horizontal Translation

Suppose *h* is a positive number



Vertical Translation

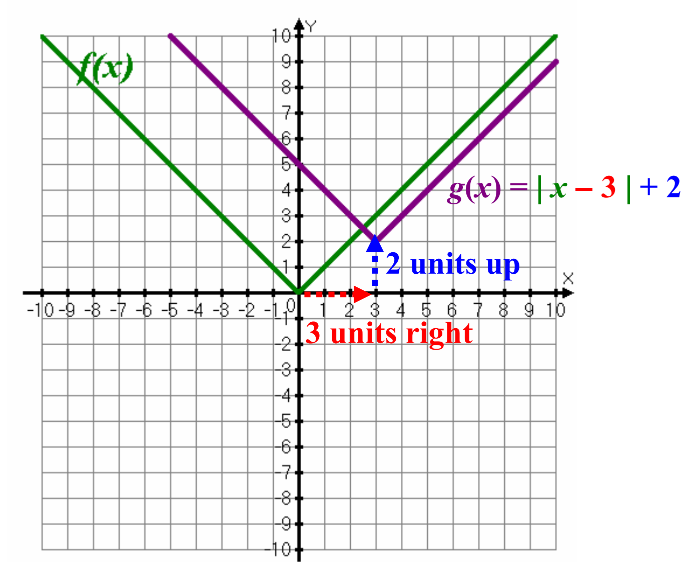
Suppose *k* is a positive number



Example 1: Describe how the graph of $g(x)$ can be obtained from $f(x)$. Then, verify by graphing both functions on the gird below.

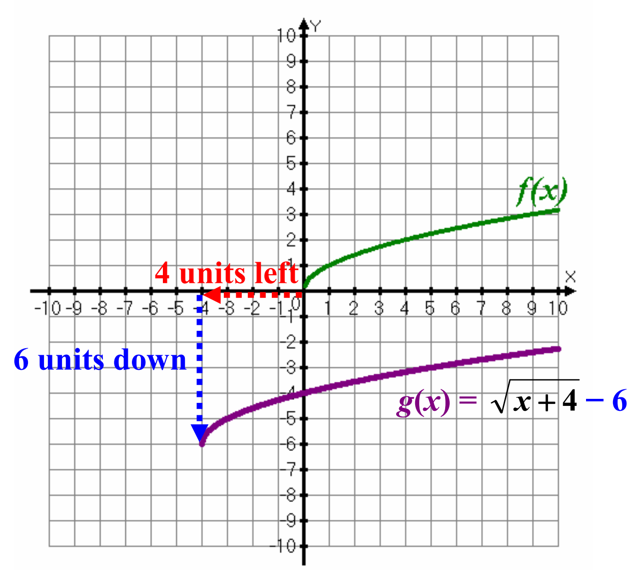
a. $f(x) = |x|$ and $g(x) = |x - 3| + 2$

To obtain $g(x)$, we take the graph of $f(x)$, **slide it 3 units right** and **2 units up.**



b. $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x+4} - 6$

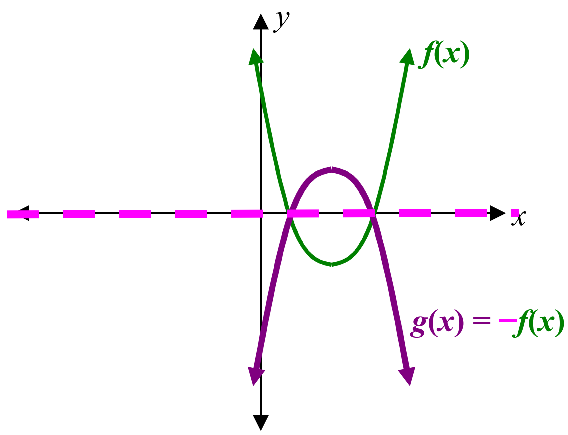
To obtain $g(x)$, we take the graph of $f(x)$, **slide it 4 units left** and **6 units down.**



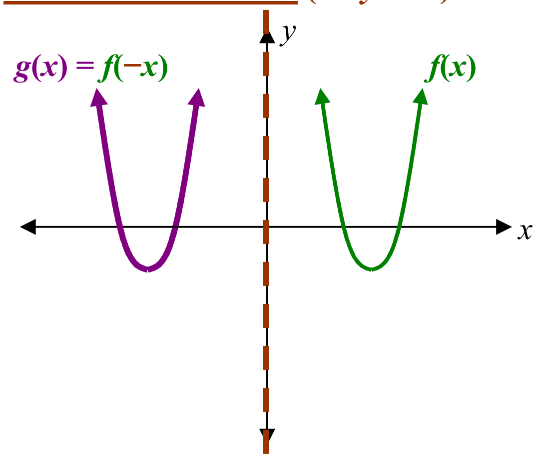
Reflection: - the transformation of a function or relation that involves obtaining a mirror image when it is flipped along the *x*-axis or *y*-axis. The **shape and size** of the original graph is **not altered**.

Reflection off the *x*-axis	**Reflection off the *y*-axis**
$g(x) = -f(x)$	$g(x) = f(-x)$
All values of *y* has to switch signs but all values of *x* remain unchanged.	**All values of *x* has to switch signs but all values of *y* remain unchanged.**

Vertical Reflection (off *x*-axis)



Horizontal Reflection (off *y*-axis)

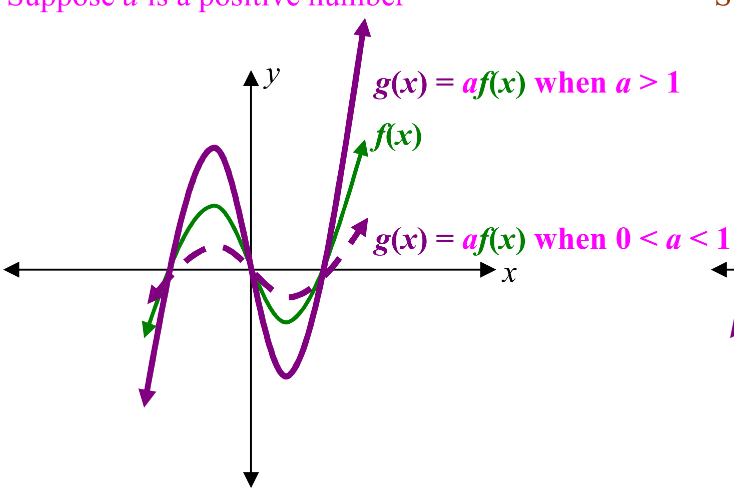


Stretching and Shrinking: - the transformation of a function or relation that involves multiplying the *x*- or *y*-values by a common factor. The **shape** of the original graph is **not altered**, but its **size is altered**.

Vertical Stretching and Shrinking	**Horizontal Stretching and Shrinking**
$g(x) = af(x)$	$g(x) = f(bx)$
***a* is the Vertical Stretch Factor**	***b* is the Horizontal Stretch Factor**
$a > 1$ **(Stretches Vertically by a factor of *a*)**	$0 < b < 1$ **(Stretches Horizontally by a factor of 1/*b*)**
$0 < a < 1$ **(Shrinks Vertically by a factor of *a*)**	$b > 1$ **(Shrinks Horizontally by a factor of 1/*b*)**

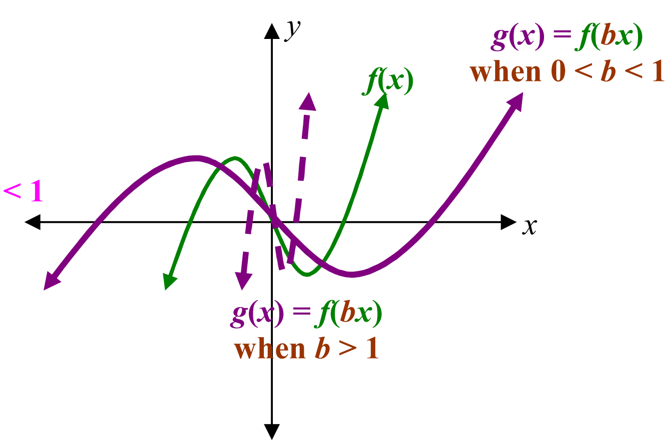
Vertical Stretching and Shrinking

Suppose *a* is a positive number



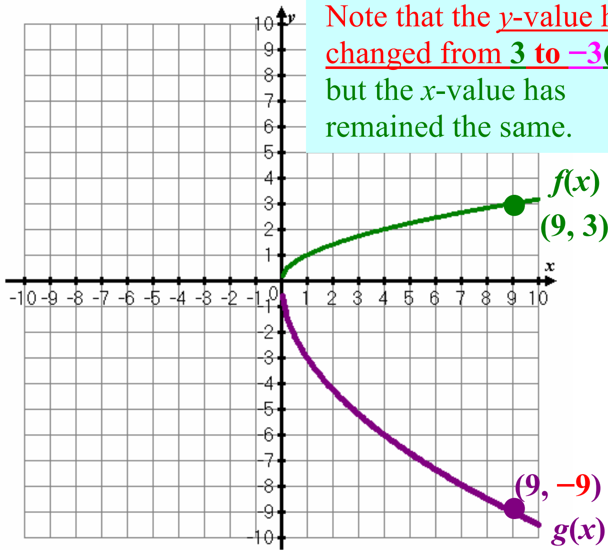
Horizontal Stretching and Shrinking

Suppose *b* is a positive number



Example 2: The graphs of $f(x)$ and $g(x)$ are given. Find the equation for function g.

a.

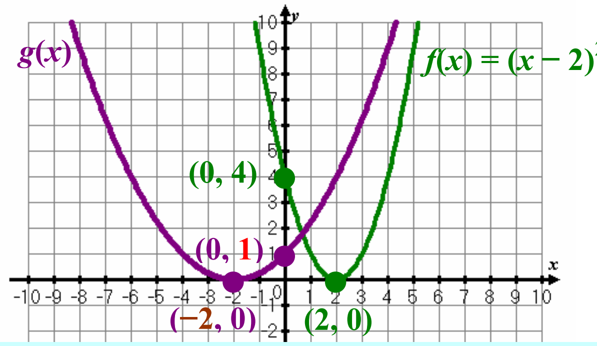


Note that the y-value has changed from **3 to −3(3)** but the x-value has remained the same.

The $f(x)$ graph had **reflected off the x-axis** and **stretched vertically by a factor of 3** to become the graph of $g(x)$.

$$g(x) = -3\sqrt{x}$$

b.



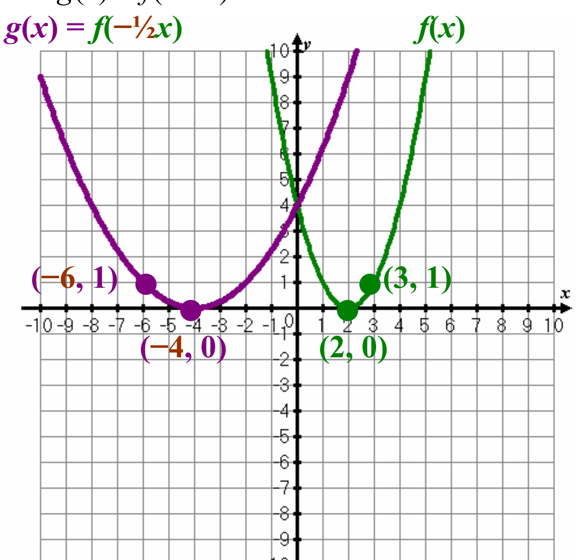
Note that the x-value has changed from **2 to −(2) at the vertex** but their y-values have remained the same. It is not a horizontal stretch **because otherwise the x-value of the vertex will be other multiples as well**. Hence, it must be a vertical shrink. Note the y-intercept has changed from **4 to ¼ (1)**.

The $f(x)$ graph had **reflected off the y-axis** and **shrinked vertically by a factor of 4** to become the graph of $g(x)$.

$$g(x) = \frac{1}{4}(-x - 2)^2$$

Example 3: Given $g(x)$ and the graph of $f(x)$ below, sketch the graph for $g(x)$. Label any important points

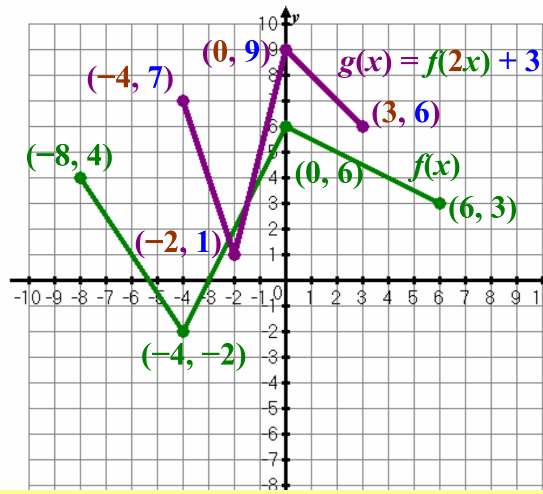
a. $g(x) = f(-\tfrac{1}{2}x)$



The $f(x)$ graph has to **reflect off the y-axis** and **stretch horizontally by a factor of 2** to become the graph of $g(x) = f(-\tfrac{1}{2}x)$.

$(2, 0) \rightarrow (-2(2), 0) \rightarrow (-4, 0)$
$(3, 1) \rightarrow (-2(3), 1) \rightarrow (-6, 1)$

b. $g(x) = f(2x) + 3$



The $f(x)$ graph has to **shrink horizontally by a factor of 2** and **translate up by 3 units** to become the graph of $g(x) = f(2x) + 3$.

$(-8, 4) \rightarrow (\tfrac{1}{2}(-8), (4) + 3) \rightarrow (-4, 7)$
$(-4, -2) \rightarrow (\tfrac{1}{2}(-4), (-2) + 3) \rightarrow (-2, 1)$
$(0, 6) \rightarrow (\tfrac{1}{2}(0), (6) + 3) \rightarrow (0, 9)$
$(6, 3) \rightarrow (\tfrac{1}{2}(6), (3) + 3) \rightarrow (3, 6)$

Example 4: Describe the transformation from $f(x)$ to $g(x)$.

a. $g(x) = -2f(x) + 3$

$g(x) = -2f(x) + 3$

- -2: **Vertically Reflected off x-axis and Vertically Stretched by a factor of 2**
- $+3$: **Vertically Translated UP 3 units**

b. $g(x) = f(-3x - 2)$

$g(x) = f(-3x - 2)$

- -3: **Horizontally Reflected off y-axis and Horizontally Shrunk by a factor of 3**
- -2: **Horizontally Translated RIGHT 2 units**

c. $g(x) = ½ f(x - 5) - 1$

$g(x) = ½ f(x - 5) - 1$

- $½$: **Vertically Shrunk by a factor of 2**
- -5: **Horizontally Translated RIGHT 5 units**
- -1: **Vertically Translated DOWN 1 unit**

Example 5: A function $f(x)$ is given, and the indicated transformations have taken place to generate $g(x)$. From the descriptions given, write the equation for $g(x)$.

a. $f(x) = x^3$; shifted right by 4 units, reflected off the x-axis and shrunk horizontally by a factor of 3.

- **First, shifted right by 4 units.** $f(x) \rightarrow f(x - 4)$
- **Then, reflected off the x-axis (Vertical Reflection).** $f(x - 4) \rightarrow -f(x - 4)$
- **Finally, shrunk horizontally by a factor of 3** $-f(x - 4) \rightarrow -f(3x - 4)$

$$g(x) = -f(3x - 4) = -(3x - 4)^3$$

b. $f(x) = |\,x\,|$; reflected off the y-axis, stretched vertically by a factor of 2, and shifted down by 5 units.

- **First, reflected off the y-axis (Horizontal Reflection)** $f(x) \rightarrow f(-x)$
- **Then, stretched vertically by a factor of 2.** $f(-x) \rightarrow 2f(-x)$
- **Finally, shifted down by 5 units.** $2f(-x) \rightarrow 2f(-x) - 5$

$$g(x) = 2f(-x) - 5 = 2|-x| - 5$$

3-4 (Part 1) Assignment: pg. 250−251 #1 to 10 (all), 11, 13, 15, 17, 19, 27, 29

3-4 Transformations of Functions (Part 2)

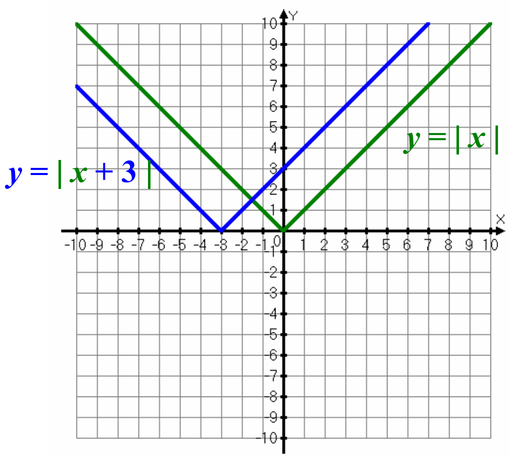
Combining Transformations:

1. **Determine the "*Parent or Standard*" function (Even Powers like x^2 and x^4; Odd Powers like x^3 and x^5; Root like $\sqrt{x}$ and $\sqrt[3]{x}$; or Absolute Value $|x|$) and sketch their graphs.**
2. **Just like the *Order of Operations*, we perform the transformation step by step in that order until we graph the given function.**

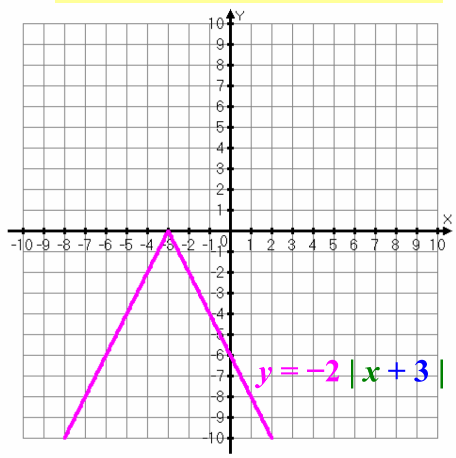
Example 1: Sketch the graph of the function, not by plotting points, but by starting with the graph of the standard functions and by applying the necessary transformation.

a. $f(x) = 5 - 2\,|\,x + 3\,|$

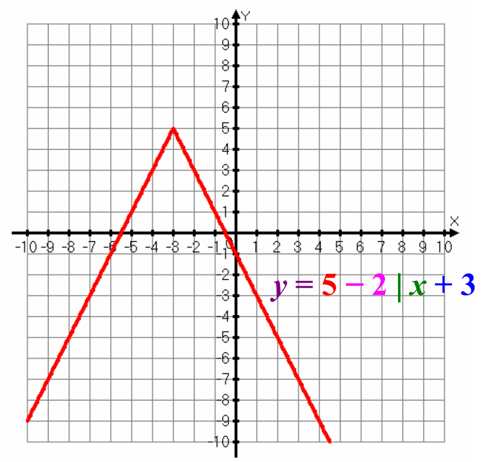
Standard Function: $y = |\,x\,|$
First, **translation of 3 units Left**



Then, **vertically reflected** and **vertically stretched** by a factor of 2

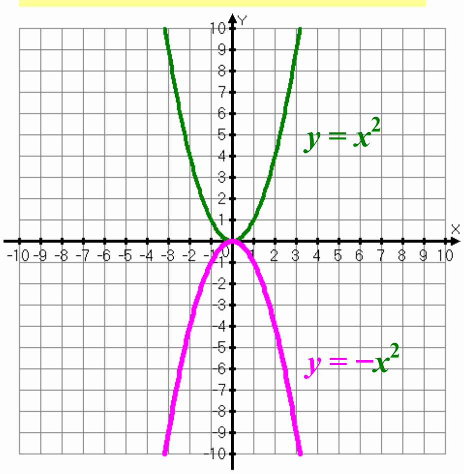


Finally, **translation by 5 units Up**

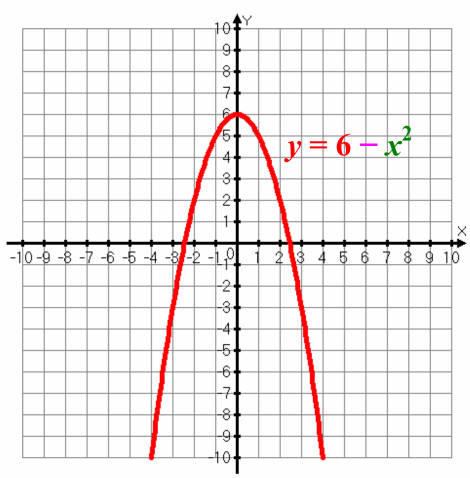


b. $f(x) = |\,6 - x^2\,|$

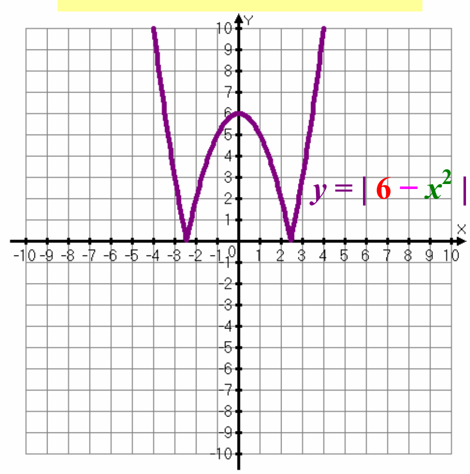
Standard Function: $y = x^2$
First, **vertically reflected**



Then, **translation by 6 units Up**



Finally, **apply Absolute Value operation to the *y*-values** ***(all negative y-values become positive)***



Even Function: - a function where $f(x) = f(-x)$ for all values of x in its domain.
- function that has **y-axis as its line of symmetry**.

Odd Function: - a function where $f(-x) = -f(x)$ for all values of x in its domain.
- has a **rotation of 180° about the origin (AFTER reflection off the y-axis (Horizontal Reflection), follows by reflection off the x-axis (Vertical Reflection)).** ***OR*** - function that its **symmetrical to *the origin***.

Example 2: For the following functions, test whether the function is odd, even or neither.

a. $f(x) = x^2 + 4$

Test for Even Function:

$f(-x) = (-x)^2 + 4$

$f(-x) = x^2 + 4$

$f(-x) = f(x)$ **It is an Even function.**

Test for Odd Function:

$-f(x) = -(x^2 + 4)$

$-f(x) = -x^2 - 4$

$-f(x) \neq f(-x)$ $\quad (-x^2 - 4) \neq (x^2 + 4)$

It is NOT an Odd function.

b. $f(x) = |-2x|$

Test for Even Function:

$f(-x) = |-2(-x)|$

$f(-x) = |2x|$

$f(-x) = f(x)$ $\quad f(x) = |-2x| = |2x|$

It is an Even function.

Test for Odd Function:

$-f(x) = -|-2x| = -|2x|$

$-f(x) \neq f(-x)$ $\quad -|2x| \neq |2x|$

It is NOT an Odd function.

c. $f(x) = \sqrt{x+2}$

Test for Even Function:

$f(-x) = \sqrt{(-x)+2} = \sqrt{-x+2}$

$f(-x) \neq f(x)$ **It is NOT an Even function.**

Test for Odd Function:

$-f(x) = -\sqrt{x+2}$

$-f(x) \neq f(-x)$ $\quad -\sqrt{x+2} \neq \sqrt{-x+2}$

It is NOT an Odd function.

d. $f(x) = \dfrac{3}{x}$

Test for Even Function:

$f(-x) = \dfrac{3}{(-x)} = -\dfrac{3}{x}$

$f(-x) \neq f(x)$ **It is NOT an Even function.**

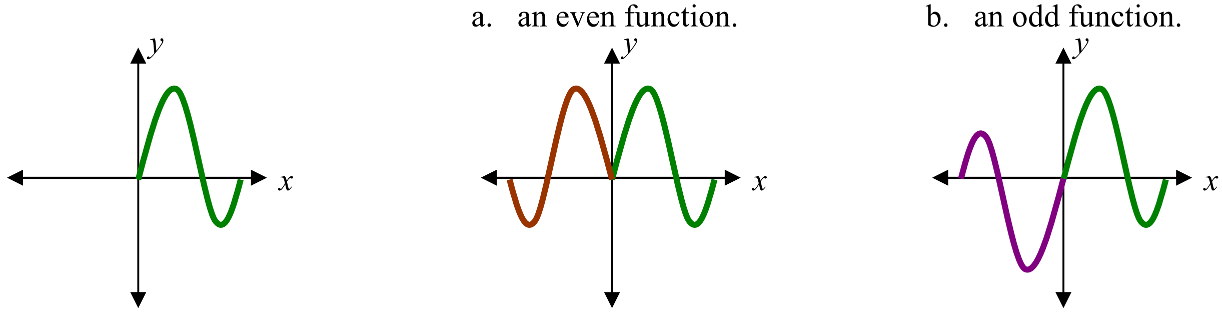
Test for Odd Function:

$-f(x) = -\left(\dfrac{3}{x}\right) = -\dfrac{3}{x}$

$-f(x) = f(-x)$ $\quad -\dfrac{3}{x} = -\dfrac{3}{x}$ **It is an Odd function.**

Example 3: The graph of a function at $x \geq 0$ is given. Complete the graph for $x < 0$ so that it will be

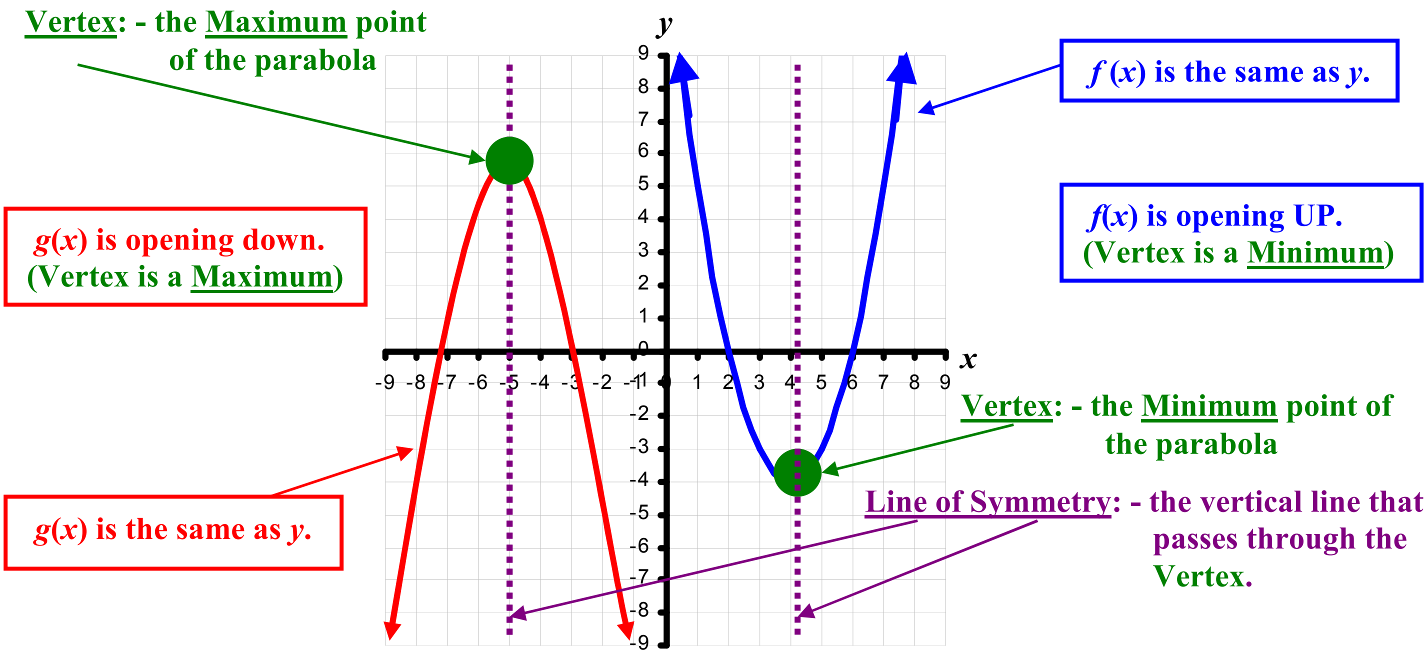
a. an even function. b. an odd function.



3-4 (Part 2) Assignment: pg. 251−253 #31, 33, 39, 43, 45, 53, 55, 69, 70;
Honours: #61, 65, 71, 73

3-5 Quadratic Functions: Maxima and Minima

Quadratic Function: - a second degree polynomial function. **(General Form: $f(x) = ax^2 + bx + c$)**
- characterized by the shape of a **parabola** when graphed.
- a parabola has a **Vertex** and **Line of Symmetry**.



For Quadratic Functions in Standard Form of $f(x) = a(x - h)^2 + k$

Vertex at (h, k) **Axis of Symmetry at $x = h$** **Domain: $x \in R$**

a = Vertical Stretch Factor

$a > 0$	Vertex at Minimum (Parabola opens UP)	Range:	$y \geq k$ (Minimum)
$a < 0$	Vertex at Maximum (Parabola opens DOWN)	Range:	$y \leq k$ (Maximum)
$\lvert a \rvert > 1$	Stretched out Vertically	$\lvert a \rvert < 1$	Shrunken in Vertically

h = Horizontal Translation (Note the standard form has $x - h$ in the bracket!)

$h > 0$	Translated Right	$h < 0$	Translated Left

k = Vertical Translation

$k > 0$	Translated Up	$k < 0$	Translated Down

For Quadratic Functions in General Form: $f(x) = ax^2 + bx + c$

y-intercept at $(0, c)$ by letting $x = 0$ (Note: Complete the Square to change to *Standard Form*)

x-intercepts at $\left(\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}, 0\right)$ if $b^2 - 4ac \geq 0$. No x-intercepts when $b^2 - 4ac < 0$

Vertex locates at $x = -\dfrac{b}{2a}$ $y = f\left(-\dfrac{b}{2a}\right)$ Minimum when $a > 0$; Maximum when $a < 0$

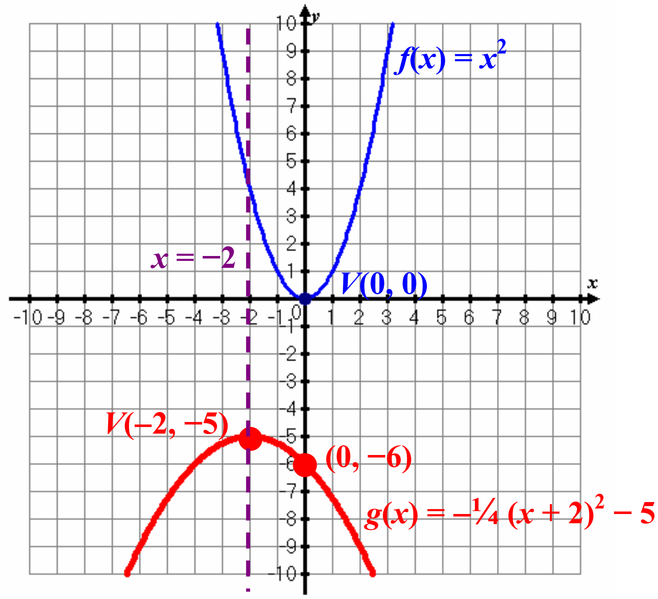
Example 1: Graph the following functions on the same grid. Describe what had happened to $g(x)$ compared to $f(x) = x^2$. Find the coordinates of the vertex and the equation of the line of symmetry for each function. State the maximum or minimum of the functions and the directions of their opening. Indicate the domain and range. Label or state the y-intercept.

a. $f(x) = x^2$ and $g(x) = -\frac{1}{4}(x + 2)^2 - 5$

$f(x) = x^2$	$g(x) = -\frac{1}{4}(x + 2)^2 - 5$
Vertex at (0, 0)	**Vertex at (−2, −5)**
Axis of Symmetry: $x = 0$	**Axis of Symmetry: $x = -2$**
Opening Up	**Opening Down ($a < 0$)**
Minimum at 0	**Maximum at −5**
Domain: $x \in R$	**Domain: $x \in R$**
Range: $f(x) \geq 0$	**Range: $g(x) \leq -5$**

The graph is reflected off the x-axis, vertically shrunk by a factor of 4, moved 2 units left & 5 units down.

For y-intercept, let $x = 0$.
$g(0) = -\frac{1}{4}(0 + 2)^2 - 5 = -6$ **y-int at (0, −6)**



b. $f(x) = x^2$ and $g(x) = 2x^2 - 16x + 33$

Complete the square for Standard Form:

$g(x) = 2x^2 - 16x + 33$
$g(x) = 2(x^2 - 8x \qquad) + 33$
$g(x) = 2(x^2 - 8x + 16) + 33 - 32$
$g(x) = 2(x^2 - 8x + 16) + 33 - 32$
$\mathbf{g(x) = 2(x - 4)^2 + 1}$ $\qquad V(4, 1)$

Vertex Location:

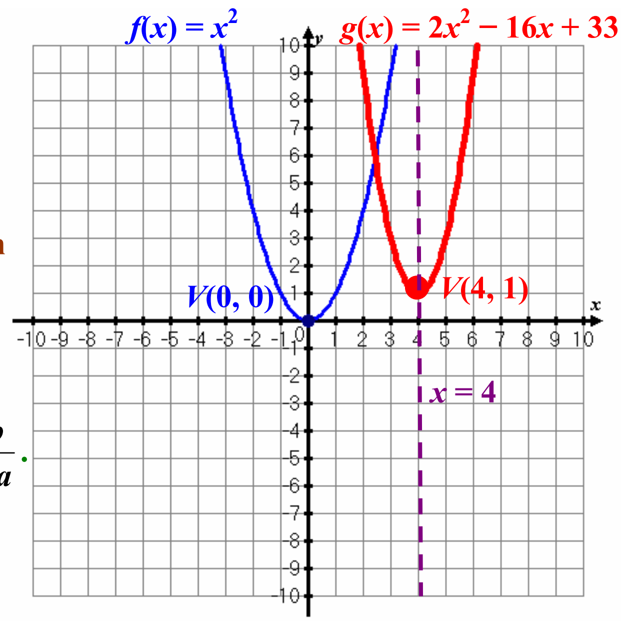
$x = -\dfrac{b}{2a} = -\dfrac{(-16)}{2(2)}$ $\qquad$ $\mathbf{x = 4}$

$y = g(4) = 2(4)^2 - 16(4) + 33$ $\qquad$ $\mathbf{y = 1}$

$g(x) = 2(x - 4)^2 + 1$

Vertex at (4, 1)
Line of Symmetry: $x = 4$
Opening Up ($a > 0$)
Minimum at 1
Domain: $x \in R$
Range: $y \geq 1$

Note that we can find the vertex either by <u>completing the square</u> OR use formula, $x = -\dfrac{b}{2a}$.



For y-intercept, let $x = 0$.
$g(0) = 2(0)^2 - 16(0) + 33 = 33$ **y-int at (0, 33)**

The graph is vertically stretched by a factor of 2, moved 4 units to the right and 1 unit up.

Example 2: Write an equation of a parabola for each given conditions.

a. Vertex at (−3, 5); $a = 2$

$y = a(x-h)^2 + k$ $V(-3, 5)$ $a = 2$

$y = (2)(x - (-3))^2 + (5)$

$$y = 2(x+3)^2 + 5$$

b. Vertex at (2, −7); opens down; congruent to $y = 3x^2$

$y = a(x-h)^2 + k$ $V(2, -7)$ $a = -3$ (from $3x^2$) (opens down)

$y = (-3)(x - (2))^2 + (-7)$

$$y = -3(x-2)^2 - 7$$

c. Vertex at (−4, 2) and passes through (−2, 6)

$y = a(x-h)^2 + k$ $V(-4, 2)$ $a = ?$

$y = a(x - (-4))^2 + (2)$

$y = a(x+4)^2 + 2$ Now to solve for a

Passes (−2, 6) ⟶ Let $x = -2$ and $y = 6$

$(6) = a((-2) + 4)^2 + 2$

$6 = 4a + 2$ $a = 1$

$$y = (x+4)^2 + 2$$

d. Vertex at (−3, −6) with y-intercept at −9

$y = a(x-h)^2 + k$ $V(-3, -6)$ $a = ?$

$y = a(x - (-3))^2 + (-6)$

$y = a(x+3)^2 - 6$ Now to solve for a

Given y-int = (0, −9) ⟶ Let $x = 0$ and $y = -9$

$(-9) = a((0) + 3)^2 - 6$

$-9 = 9a - 6$

$-3 = 9a$ $a = -\frac{1}{3}$

$$y = -\frac{1}{3}(x+3)^2 - 6$$

Example 3: One of the x-intercept of the parabola is at −5 and its vertex is at (−1, −6). Find the other x-intercept and the equation of this parabola. Graph the resulting equation.

From the location of the vertex, we can see that the **axis of symmetry is $x = -1$**. Since the distance between the symmetry axis and the first x-int is 4 units, this means the mirror point of the other x-int is at (−1 + 4, 0) = **(3, 0)**

$y = a(x-h)^2 + k$ $V(-1, -6)$ $a = ?$

$y = a(x - (-1))^2 + (-6)$

$y = a(x+1)^2 - 6$ Now to solve for a

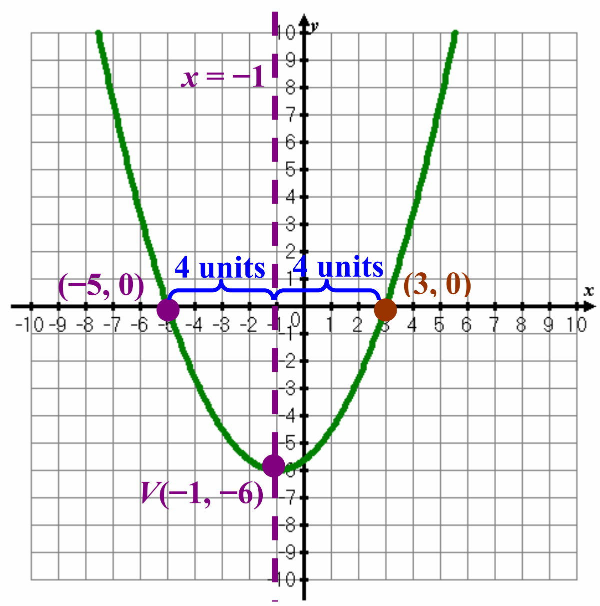
Given x-int = (−5, 0) ⟶ Let $x = -5$ and $y = 0$

$(0) = a((-5) + 1)^2 - 6$

$0 = 16a - 6$

$6 = 16a$ $a = \frac{3}{8}$

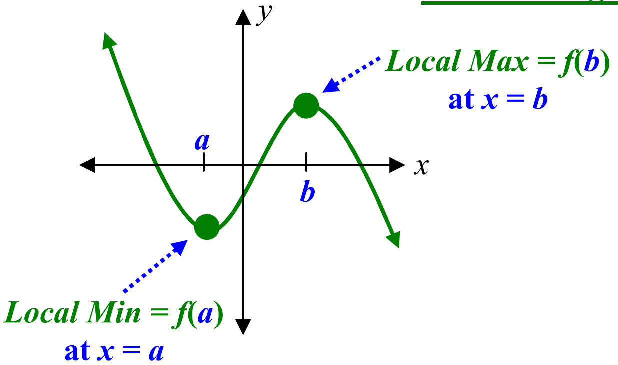
$$y = \frac{3}{8}(x+1)^2 - 6$$

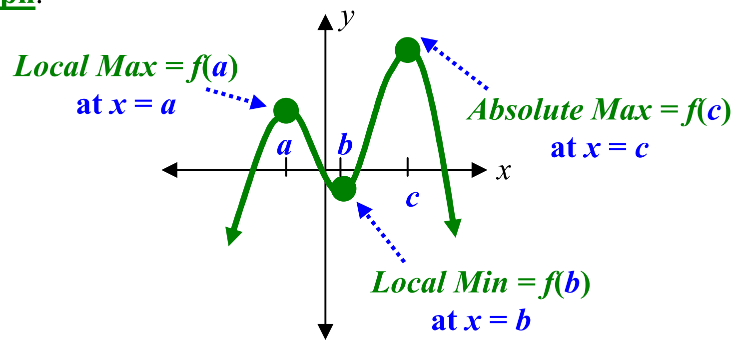


Extreme Values: - any point on a graph that indicate a maximum or a minimum y-value.

Absolute Maximum / Minimum: - an extreme value of a graph where the point indicate it has the **largest** y-value (maximum) or **smallest** y-value (minimum).

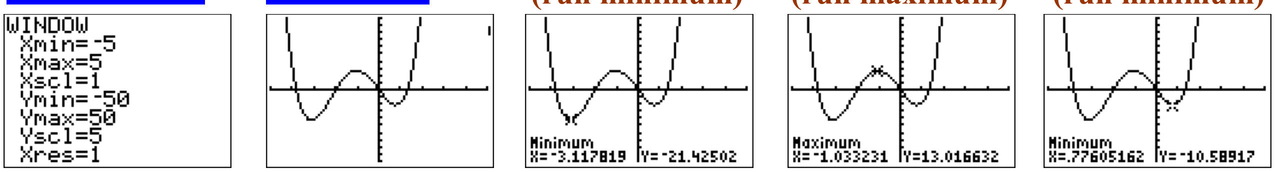
Local Maximum / Minimum: - an extreme value of a graph where the point indicate it has the **relatively large** *y*-value (maximum) or **relatively small** *y*-value (minimum) in the immediate vicinity. **Local Max / Min is *NOT* the largest or smallest *y*-value of the graph**.





Example 4: Using a graphing calculator and the WINDOW settings of [−5, 5] by [−50, 50] , find all the extreme values of $f(x) = 2x^4 + 9x^3 - 20x$ to the nearest hundredth.

WINDOW **GRAPH** **Find Absolute Min (run minimum)** **Find Local Max (run maximum)** **Find Local Min (run minimum)**



Extreme Values for $f(x) = 2x^4 + 9x^3 - 20x$

Absolute Minimum at −21.32 when $x = -3.12$

Local Maximum at 13.02 when $x = -1.03$ **and Local Minimum at −10.59** when $x = 0.78$

Maximum and Minimum Problems

When Solving Maximum and Minimum Word Problem:

1. **Draw any appropriate diagrams.**
2. **Define the Variables involved.**
3. **Determine the relationship between these variables (*y* in terms of *x*).**
4. **Write the Maximum or Minimum Function using ONE variable. Expand if necessary.**
5. **Find the Vertex, (*x*, Min) or (*x*, Max), of the quadratic function by Completing the Square, using the Formula** $x = -\frac{b}{2a}$ **with** $y = f\left(-\frac{b}{2a}\right)$, **or by Graphing.**
6. **Report the answer in a complete sentence.**

Example 5: Find two numbers have a difference of 9 and their product is a minimum.

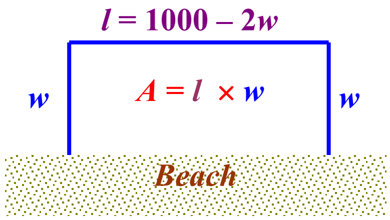
Let x = larger of the two numbers
$(x - 9)$ = smaller of the two numbers
y = product

$y = x(x - 9)$
$y = x^2 - 9x$
$y = (x^2 - 9x \quad)$ Complete the square to find Min and Vertex
$y = (x^2 - 9x + (\tfrac{9}{2})^2) - \tfrac{81}{4}$ $\quad y = (x - \tfrac{9}{2})^2 - \tfrac{81}{4}$ $\quad V(\tfrac{9}{2}, -\tfrac{81}{4})$

From $V(x, Min\ y)$,
$x = \tfrac{9}{2}$ **and** $(x - 9) = -\tfrac{9}{2}$

The numbers are $\tfrac{9}{2}$ and $-\tfrac{9}{2}$, and Min Product = $-\tfrac{81}{4}$

Example 6: A 1000 metre rope is used to section off a rectangular swimming area by the beach. Determine the dimensions of the rectangle that will yield a maximum area.



Let w = width l = length A = area

***Perimeter* = 1000**

$2w + l = 1000$ (one side is bounded by the beach)

$l = (1000 - 2w)$

$A = l \times w$

$A = (1000 - 2w)w$

$A = -2w^2 + 1000w$ (Use Formula to find w at Max Area)

$l = 1000 - 2w$

$l = 1000 - 2(250)$

Vertex locates at $w = -\dfrac{b}{2a} = -\dfrac{1000}{2(-2)}$

Max Area $= -2(250)^2 + 1000(250)$

w = 250 m **l = 500 m**

Max Area = 125,000 m²

Example 7: A marketing firm for the Santa Clara Transit has determined that there will be 7,500 less people riding the light rail for every five cents increased on the fare. There are currently 300,000 people ride the light rail at $1.50 on a daily basis. At what price should the Santa Clara Transit charge per fare to yield the maximum revenue?

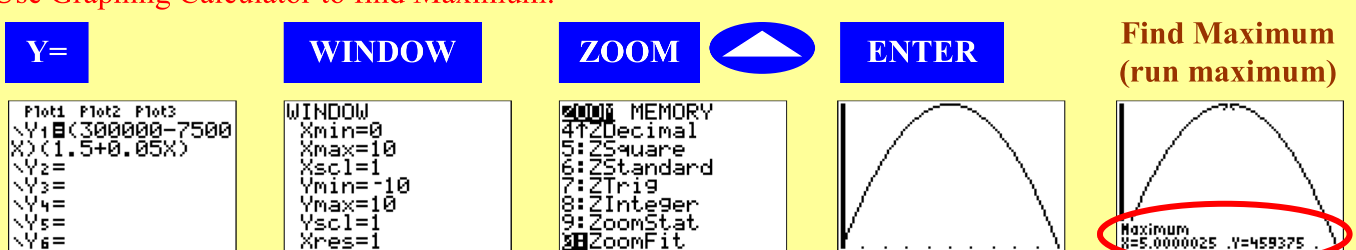
Revenue = (Number of Riders)(Price per Fare)

Let x = Each Five-cents Increase on Price per Fare

R = Revenue

$R = (300{,}000 - 7500x)(1.50 + 0.05x)$ [Every 5-cents or $0.05x$ increase on $1.50, there will be 7500 less riders from 300,000 current riders]

Use Graphing Calculator to find Maximum:



Change X_{min} to 0
(x should be greater than 0)

At Maximum, x = 5. This means that the increase is (5)($0.05) = $0.25.
Hence the new fare should be $1.50 + $0.25.
Max Revenue = (300,000 − 7500(5))(1.50 + 0.05(5))

New Fare = $1.75
Max Revenue = $459,375

3-5 Assignment: pg. 261−263 #3, 9, 15, 23, 31, 35, 37, 39, 43, 49, 55, 61;
Honours: #65, 67, 71b and 71c

3-6 Combining Functions

Functions can be combined in many ways just as numbers can be combined through various operations.

Operation	***Notation***	***Domain*** (Let F = Domain of $f(x)$ and G = Domain of $g(x)$)
Addition	$(f+g)(x) = f(x) + g(x)$	$F \cap G$ (F intersects G)
Subtraction	$(f-g)(x) = f(x) - g(x)$	$F \cap G$ (F intersects G)
Multiplication	$(fg)(x) = f(x)\,g(x)$	$F \cap G$ (F intersects G)
Division	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	$\{x \in F \cap G \mid g(x) \neq 0\}$

Example 1: For $f(x) = 3\sqrt{x} - 6$ and $g(x) = \dfrac{2}{x-4}$ and, find and evaluate the following. State the domain for each case. Graph each of the following to verify your answer.

a. $(f+g)(x)$ and $(f+g)(1)$

$(f+g)(x) = f(x) + g(x)$

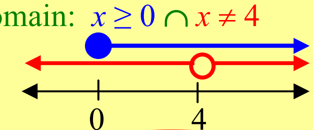
$$\mathbf{(f+g)(x) = 3\sqrt{x} - 6 + \frac{2}{x-4}}$$

$(f+g)(1) = 3\sqrt{(1)} - 6 + \dfrac{2}{(1)-4}$

$(f+g)(1) = 3 - 6 + \dfrac{2}{-3}$

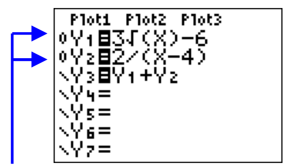
$$\mathbf{(f+g)(1) = -\frac{11}{3}}$$

Domain: $x \geq 0 \cap x \neq 4$



$\{x \mid x \geq 0$ and $x \neq 4\}$ or $[0, 4) \cup (4, \infty)$

Y=

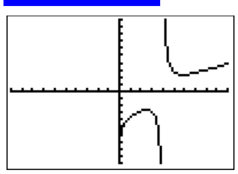


Take cursor to the space before Y_1 and press ENTER repeatedly until "**o**" symbol appears.

To enter Y_1 & Y_2, use **VARS**

(see Ex. 4 of 3-1)

GRAPH



b. $(f-g)(x)$ and $(f-g)(0)$

$(f-g)(x) = f(x) - g(x)$

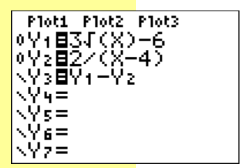
$(f-g)(x) = (3\sqrt{x} - 6) - \left(\dfrac{2}{x-4}\right)$

$$\mathbf{(f-g)(x) = 3\sqrt{x} - 6 - \frac{2}{x-4}}$$

$(f-g)(0) = 3\sqrt{(0)} - 6 - \dfrac{2}{(0)-4}$

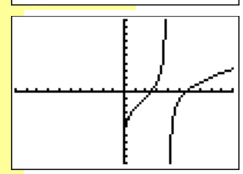
$(f-g)(0) = 0 - 6 - \dfrac{2}{-4}$

$$\mathbf{(f-g)(0) = -\frac{11}{2}}$$



Domain: $x \geq 0 \cap x \neq 4$

$\{x \mid x \geq 0$ and $x \neq 4\}$ or $[0, 4) \cup (4, \infty)$



c. $(fg)(x)$ and $(fg)(-2)$

$(fg)(x) = f(x)\,g(x)$

$(fg)(x) = (3\sqrt{x} - 6)\left(\dfrac{2}{x-4}\right)$

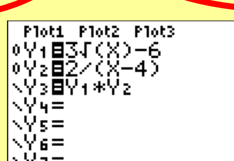
$$\mathbf{(fg)(x) = \frac{6\sqrt{x} - 12}{x-4}}$$

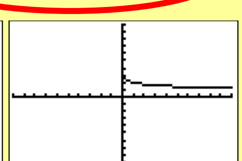
$(fg)(-2) = \dfrac{6\sqrt{(-2)} - 12}{(-2)-4}$

$(fg)(-2) = \dfrac{6i\sqrt{2} - 12}{-6}$

$$\mathbf{(fg)(-2) = -i\sqrt{2} + 2}$$

Domain: $x \geq 0 \cap x \neq 4$





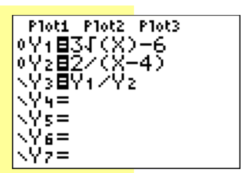
$\{x \mid x \geq 0$ and $x \neq 4\}$ or $[0, 4) \cup (4, \infty)$

d. $\left(\frac{f}{g}\right)(x)$

$\left(\frac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{3\sqrt{x} - 6}{\left(\frac{2}{x-4}\right)}$

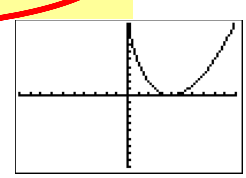
$\left(\frac{f}{g}\right)(x) = \dfrac{(3\sqrt{x} - 6)(x-4)}{2}$

$$\mathbf{\left(\frac{f}{g}\right)(x) = \frac{3x\sqrt{x} - 12\sqrt{x} - 6x + 24}{2}}$$

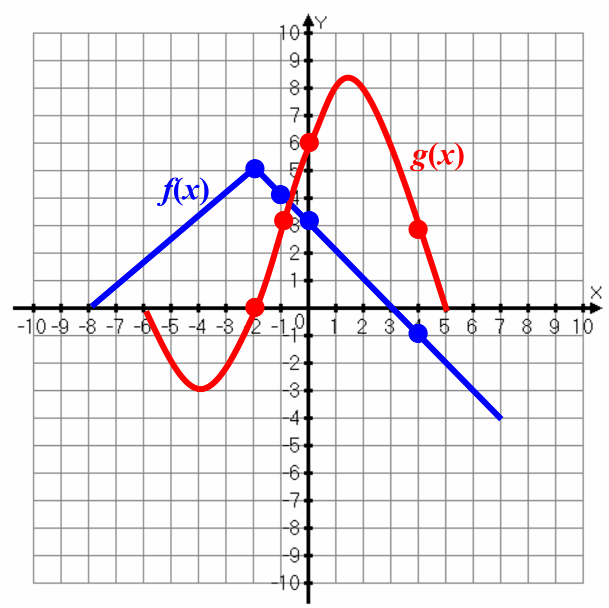


Domain: $x \geq 0 \cap x \neq 4$

$\{x \mid x \geq 0$ and $x \neq 4\}$ or $[0, 4) \cup (4, \infty)$



Example 2: The graphs of the functions $f(x)$ and $g(x)$ is shown below. Evaluate the following.



a. $(f+g)(0)$

$(f+g)(0) = f(0) + g(0)$
$(f+g)(0) = (3) + (6)$

$\mathbf{(f+g)(0) = 9}$

b. $(f-g)(4)$

$(f-g)(4) = f(4) - g(4)$
$(f-g)(4) = (-1) - (3)$

$\mathbf{(f-g)(4) = -4}$

c. $(fg)(-1)$

$(fg)(-1) = f(-1) \times g(-1)$
$(fg)(-1) = (4) \times (3)$

$\mathbf{(fg)(-1) = 12}$

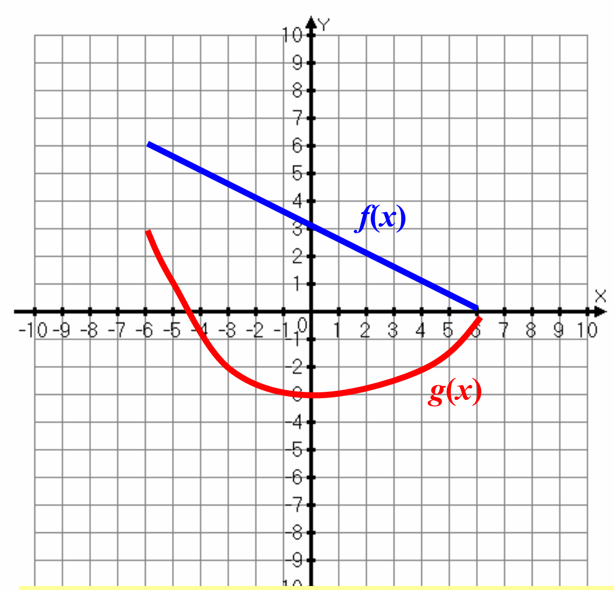
d. $\left(\frac{f}{g}\right)(-2)$

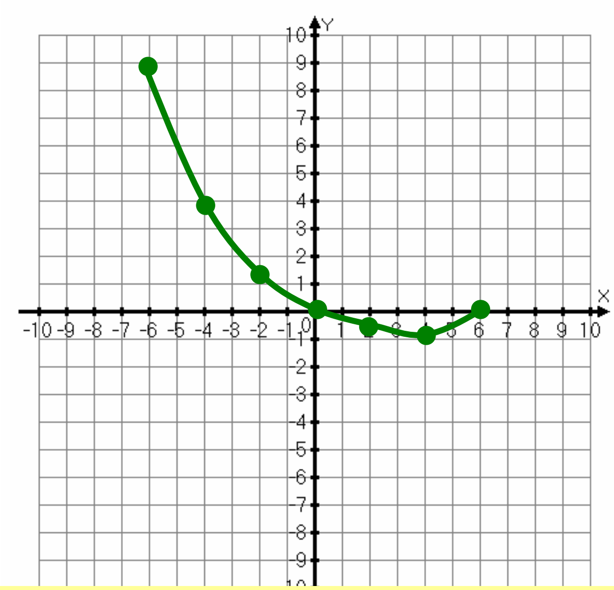
$\left(\frac{f}{g}\right)(-2) = \frac{f(-2)}{g(-2)}$

$\left(\frac{f}{g}\right)(-2) = \frac{(5)}{(0)}$

$\left(\frac{f}{g}\right)(-2) =$ **undefined**

Example 3: The graphs of the functions $f(x)$ and $g(x)$ is shown below. Use graphical addition to graph $(f+g)(x)$.





We can add the two functions point by point.

$(f+g)(-6) = f(-6) + g(-6) = (6) + (3) = \mathbf{9}$
$(f+g)(-4) = f(-4) + g(-4) = (5) + (-1) = \mathbf{4}$
$(f+g)(-2) = f(-2) + g(-2) = (4) + (-2.5) = \mathbf{1.5}$
$(f+g)(0) = f(0) + g(0) = (3) + (-3) = \mathbf{0}$
$(f+g)(2) = f(2) + g(2) = (2) + (-2.7) = \mathbf{-0.7}$
$(f+g)(4) = f(4) + g(4) = (1) + (-2) = \mathbf{-1}$
$(f+g)(6) = f(6) + g(6) = (0) + (0) = \mathbf{0}$

<u>Composition Function</u>: - when one function is substituted into the variables of another function.

$(f \circ g)(x) = f(g(x))$ (Read as "f of g of x")

Domain of Composite Functions: Include BOTH Domains of functions f and g.

Always Evaluate the domain of the <u>INNER Bracket Function</u> first!

Example 4: For $p(x) = 2x^2 - 3$ and $q(x) = 3x + 1$, find the following and state the domain.

a. $(p \circ q)(x)$ and $(p \circ q)(-4)$

$(p \circ q)(x) = p(q(x))$

$(p \circ q)(x) = 2(3x + 1)^2 - 3$

$(p \circ q)(x) = 2(9x^2 + 6x + 1) - 3$

$(p \circ q)(x) = 18x^2 + 12x + 2 - 3$

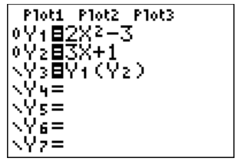
$\mathbf{(p \circ q)(x) = 18x^2 + 12x - 1}$

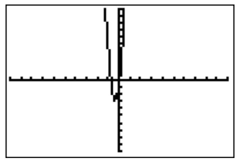
$(p \circ q)(-4) = 18(-4)^2 + 12(-4) - 1$

$\mathbf{(p \circ q)(-4) = 239}$

Domain: $x \in R \cap x \in R$

$\{x \mid x \in R\}$





b. $(q \circ p)(x)$ and find x when $(q \circ p)(x) = 16$

$(q \circ p)(x) = q(p(x))$

$(q \circ p)(x) = 3(2x^2 - 3) + 1$

$(q \circ p)(x) = 6x^2 - 9 + 1$

$\mathbf{(q \circ p)(x) = 6x^2 - 8}$

$(q \circ p)(x) = 16$

$6x^2 - 8 = 16$

$6x^2 = 16 + 8$

$6x^2 = 24$

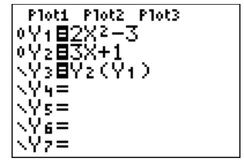
$x^2 = \frac{24}{6}$

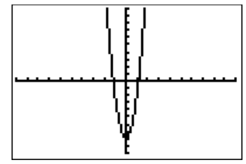
$x^2 = 4$

$\mathbf{x = \pm 2}$

Domain: $x \in R \cap x \in R$

$\{x \mid x \in R\}$





Example 5: For $g(x) = \sqrt{3 - x}$, find $(g \circ g)(x)$ and $(g \circ g)(-1)$. State the domain.

$(g \circ g)(x) = g(g(x))$

$\mathbf{(g \circ g)(x) = \sqrt{3 - \sqrt{3 - x}}}$

$(g \circ g)(-1) = \sqrt{3 - \sqrt{3 - (-1)}}$

$(g \circ g)(-1) = \sqrt{3 - \sqrt{4}}$

$(g \circ g)(-1) = \sqrt{3 - 2}$

$\mathbf{(g \circ g)(-1) = 1}$

Domain:

$3 - x \geq 0$ $\cap$ $3 - \sqrt{3 - x} \geq 0$

$-x \geq -3$ $\qquad -\sqrt{3 - x} \geq -3$

$x \leq 3$ $\qquad \sqrt{3 - x} \leq 3$

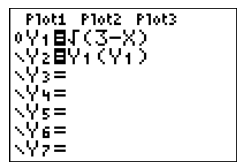
$(\sqrt{3 - x})^2 \leq 3^2$

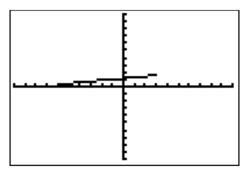
$3 - x \leq 9$

$-x \leq 6$

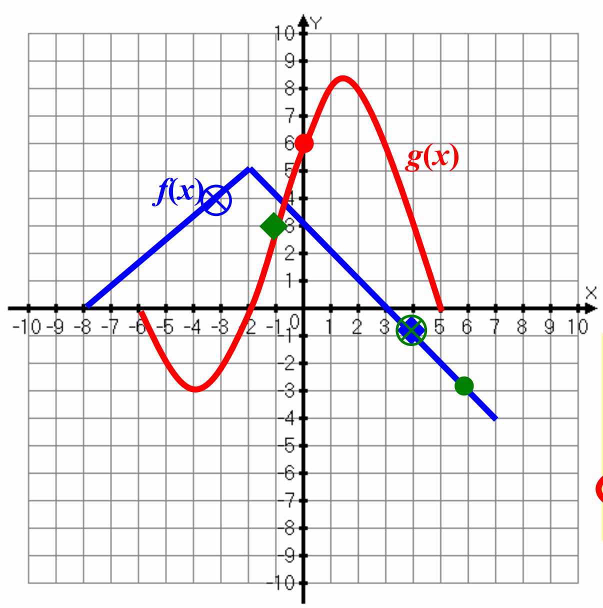
$x \geq -6$

$\{x \mid -6 \leq x \leq 3\}$





Example 6: The graphs of the functions $f(x)$ and $g(x)$ is shown below. Evaluate the following.



a. $(f \circ g)(0)$

$(f \circ g)(0) = f(g(0))$ $\quad g(0) = 6$

$(f \circ g)(0) = f(6)$

$\mathbf{(f \circ g)(0) = -3}$

b. $(g \circ f)(4)$

$(g \circ f)(4) = g(f(4))$ $\quad f(4) = -1$

$(g \circ f)(4) = g(-1)$

$\mathbf{(g \circ f)(4) = 3}$

c. $(f \circ f)(-3)$

$(f \circ f)(-3) = f(f(-3))$ $\quad f(-3) = 4$

$(f \circ f)(-3) = f(4)$

$\mathbf{(f \circ f)(-3) = -1}$

Example 7: Express the functions below in the form of $(f \circ g)(x)$.

a. $F(x) = (x + 2)^4$

$(f \circ g)(x) = f(g(x)) = (x + 2)^4$

$f(x) = x^4 \quad g(x) = x + 2$

b. $H(x) = | x^2 - 6 |$

$(f \circ g)(x) = f(g(x)) = | x^2 - 6 |$

$f(x) = | x | \quad g(x) = x^2 - 6$

or

$(f \circ g)(x) = f(g(x)) = | x^2 - 6 |$

$f(x) = | x - 6 | \quad g(x) = x^2$

c. $G(x) = \sqrt{9 - x^3}$

$(f \circ g)(x) = f(g(x)) = \sqrt{9 - x^3}$

$f(x) = \sqrt{x} \quad g(x) = 9 - x^3$

or

$f(x) = \sqrt{9 - x} \quad g(x) = x^3$

Example 8: A manufacturer sold an electronic appliance to a wholesaler with a 60% mark-up. The wholesaler then sell it to a retailer at a 80% mark-up. Finally, the retailer sell the same product to the consumer at a 140% mark-up. What is the final price of the item if it costs the manufacturer \$7.00 to build?

Note: Final Price = Original Price + Original Price × Mark-up Rate
Final Price = Original Price (1 + Mark-up Rate)

Let $m(x)$ = manufacturer's mark-up $\quad m(x) = 1.60x$
$w(x)$ = wholesaler's mark-up $\quad w(x) = 1.80x$
$r(x)$ = retailer's mark-up $\quad r(x) = 2.40x$

Price to Consumer = $(r \circ w \circ m)(x) = r(w(m(x))) = 2.40(1.80(1.60x))$
Price to Consumer = 6.912*x*

At x = \$7.00, Price to Consumer = 6.912(\$7.00)

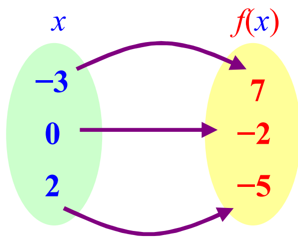
Price to Consumer = \$48.38

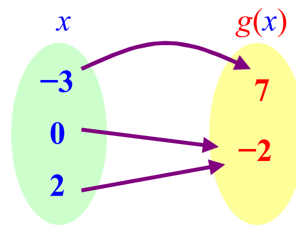
3-6 Assignment: pg. 268−271 #1, 5, 7, 11, 17, 19, 21, 23 to 28, 31, 35, 45, 49, 57, 61;
Honours: #39, 43, 51 and 65

3-7 One-to-One Functions and their Inverses

One-to-One Function: - a function where every input has a unique output.
- in essence, it **passes both Vertical Line Test** (necessary for an equation to be called a function), **and the Horizontal Line Test** (as a one-to-one function).

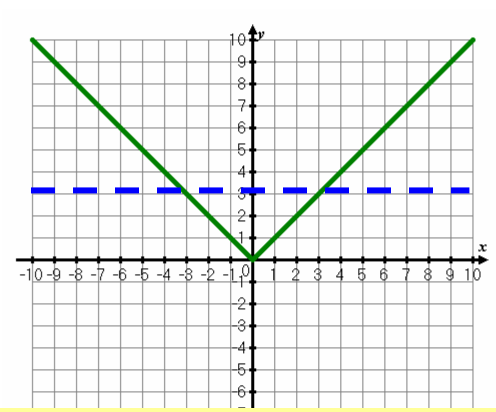
Examples: $f(x)$ is a one-to-one function $g(x)$ is not a one-to-one function





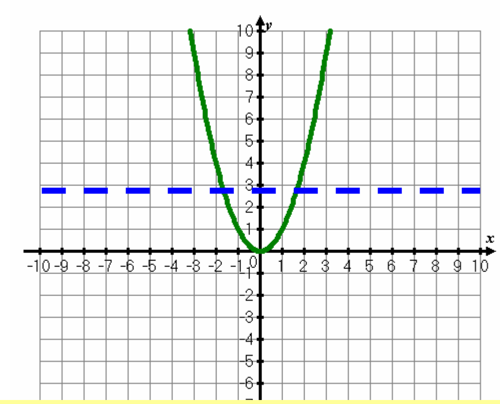
Example 1: Graph the following function and determine whether it is a one-to-one function.

a. $f(x) = |\, x \,|$



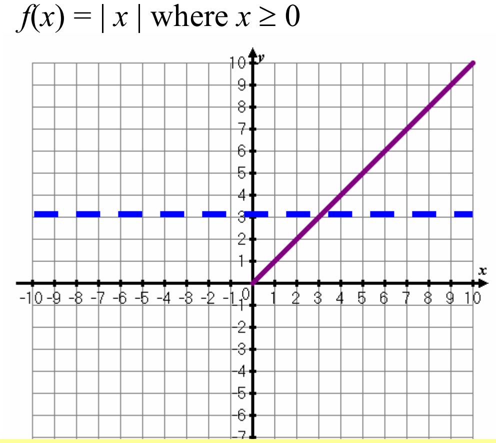
It is a function (pass the vertical line test), **but NOT a One-to-One Function, did NOT pass the horizontal line test.**

b. $f(x) = x^2$



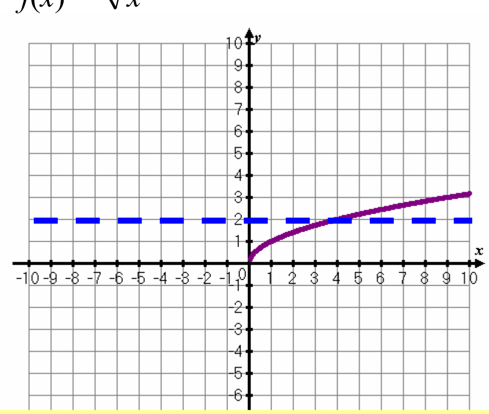
It is a function (pass the vertical line test), **but NOT a One-to-One Function, did NOT pass the horizontal line test.**

c. $f(x) = |\, x \,|$ where $x \geq 0$



It is a function (pass the vertical line test), **and it is a One-to-One Function, (pass the horizontal line test).**

d. $f(x) = \sqrt{x}$



It is a function (pass the vertical line test), **and it is a One-to-One Function, (pass the horizontal line test).**

Inverse Function $f^{-1}(x)$: - function that contains a set of order pairs that had the **x and y components (including Domain and Range) *switched* from the original One-to-One Function**.

- a **function that is NOT a One-to-One Function does NOT have an Inverse Function**.

$f(x)$ = One-to-One Function		$f^{-1}(x)$ = Inverse Function
(x, y)	⟶	(y, x)

Domain of $f(x)$ → Range of $f^{-1}(x)$

Range of $f(x)$ → Domain of $f^{-1}(x)$

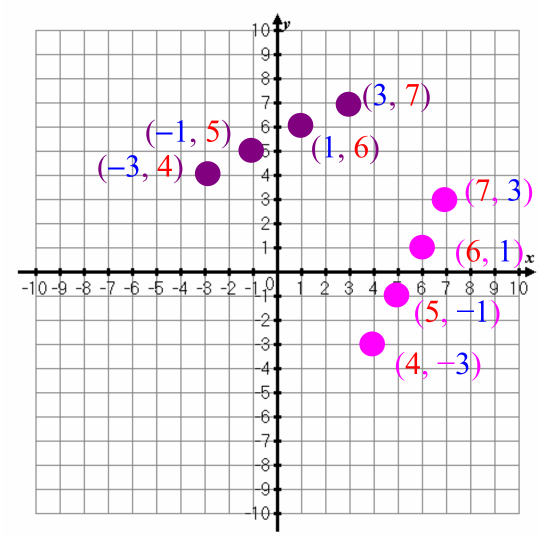
Note: $f^{-1}(x) \neq \dfrac{1}{f(x)}$ (Inverse is DIFFERENT than Reciprocal)

Example 1: Find the inverse order pairs of f = {(−3, 4), (−1, 5), (1, 6), (3, 7)}. Graph the order pairs for function f and its inverse. Determine the domain and range for both functions.

f: (−3, 4), (−1, 5), (1, 6), (3, 7)
Domain: {x = −3, −1, 1, 6}
Range: {y = 4, 5, 6, 7}

switch x and y for f^{-1}

f^{-1}: (4, −3), (5, −1), (6, 1), (7, 3)
Domain: {x = 4, 5, 6, 7}
Range: {y = −3, −1, 1, 6}

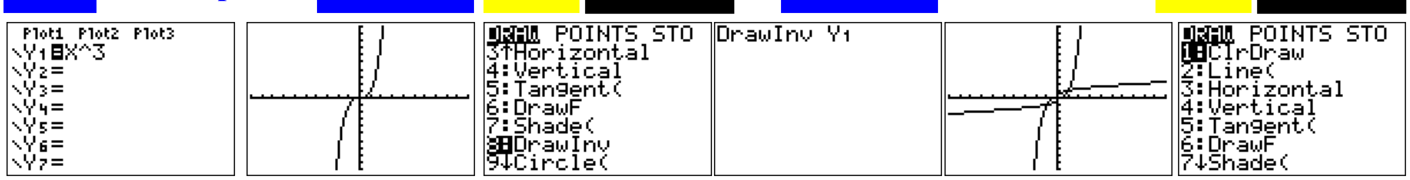


To find the Inverse Function Equation from a One-to-One Function:

1. **Replace all x with y, and y with x (or $f(x)$ with x).**
2. **Isolate the new y.**
3. **Use the Inverse Function Notation, $f^{-1}(x)$. State the Domain and Range of the Inverse Function. Recall Domain of $f(x)$ becomes Range of $f^{-1}(x)$ and Range of $f(x)$ becomes Domain of $f^{-1}(x)$.**

To Draw Inverse Functions on a Graphing Calculator:

Y= → **Enter Function and Graph** → **GRAPH** → **2nd** → **DRAW PRGM** → **ENTER** → **Select ClrDraw to erase inverse graph when finished.** → **2nd** → **DRAW PRGM**



Example 2: Find the inverse of the following functions. Graph the functions and their inverses. State the domains and ranges for both functions.

a. $f(x) = \dfrac{2x-6}{3}$

$f(x) = \dfrac{2x-6}{3}$

$y = \dfrac{2x-6}{3}$

$x = \dfrac{2y-6}{3}$ (switch x & y for inverse)

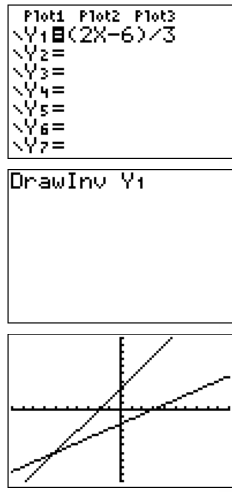
$3x = 2y - 6$ (Solve for the new y)

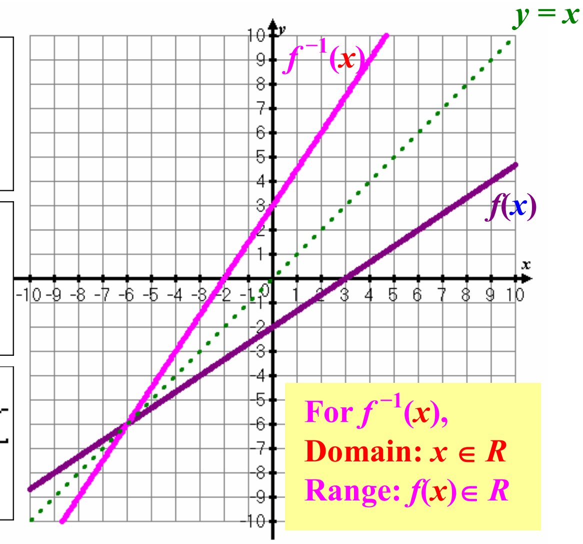
$3x + 6 = 2y$

$\dfrac{3x+6}{2} = y$

$$f^{-1}(x) = \frac{3x+6}{2}$$

For $f(x)$,
Domain: $x \in R$
Range: $f(x) \in R$





For $f^{-1}(x)$,
Domain: $x \in R$
Range: $f(x) \in R$

b. $g(x) = x^2 - 4$, $x \geq 0$

$g(x) = x^2 - 4$

$y = x^2 - 4$

$x = y^2 - 4$ (switch x & y for inverse)

$x + 4 = y^2$ (Solve for the new y)

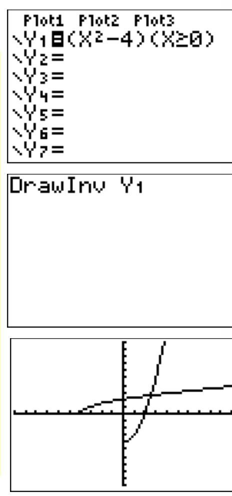
(Positive Root only since Range of $g(x) \geq 0$)

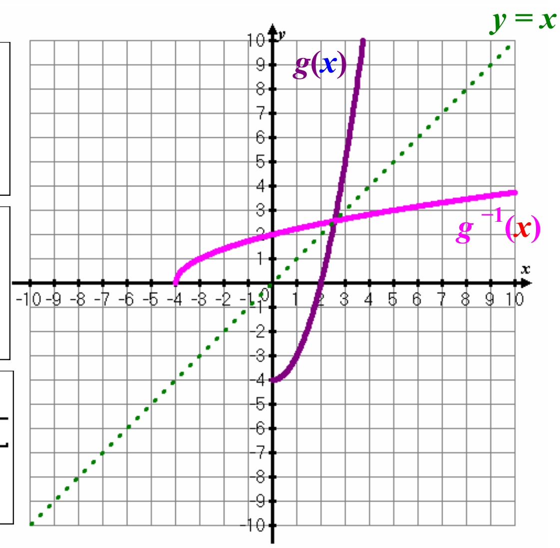
$\sqrt{x+4} = y$

$$g^{-1}(x) = \sqrt{x+4}$$

For $g(x)$,
Domain: $x \geq 0$
Range: $g(x) \geq -4$

For $g^{-1}(x)$,
Domain: $x \geq -4$
Range: $g^{-1}(x) \geq 0$





c. $h(x) = (x-4)^2$, $x \geq 4$

$h(x) = (x-4)^2$

$y = (x-4)^2$

$x = (y-4)^2$ (switch x & y for inverse)

$\sqrt{x} = y - 4$ (Solve for the new y)

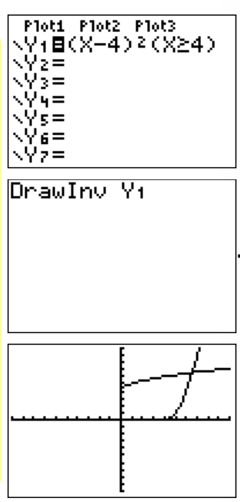
(Positive Root only since Range of $h(x) \geq 0$)

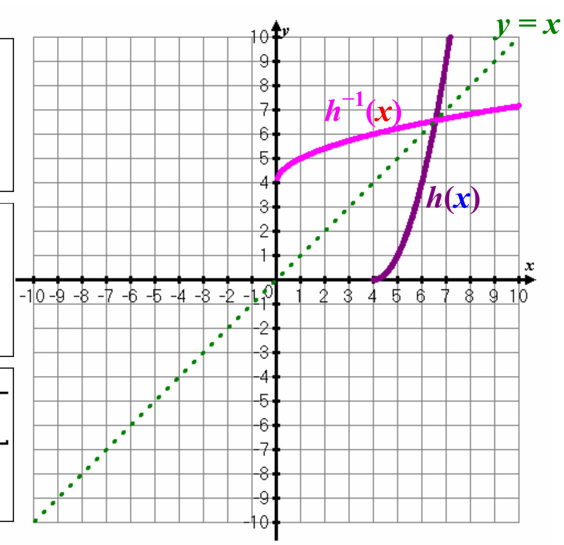
$\sqrt{x} + 4 = y$

$$h^{-1}(x) = \sqrt{x} + 4$$

For $h(x)$
Domain: $x \geq 4$
Range: $h(x) \geq 0$

For $h^{-1}(x)$,
Domain: $x \geq 0$
Range: $h^{-1}(x) \geq 4$





d. $q(x) = \sqrt{5-x}$

$q(x) = \sqrt{5-x}$

$y = \sqrt{5-x}$

$x = \sqrt{5-y}$ (switch x & y for inverse)

$x^2 = 5 - y$ (Solve for the new y)

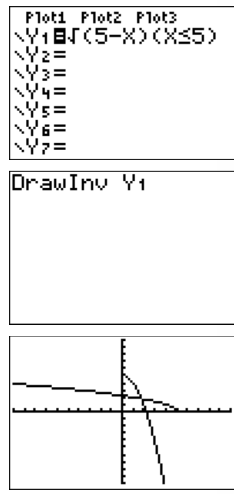
$y = -x^2 + 5$

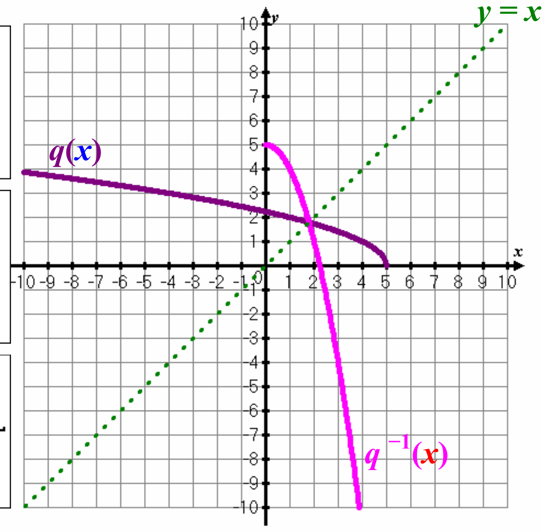
$q^{-1}(x) = -x^2 + 5$ **where** $x \geq 0$

Since $q(x)$ is a horizontal parabola with one branch only, the inverse will be a vertical parabola with only one branch as well.

For $q(x)$,
Domain: $x \leq 5$
Range: $q(x) \geq 0$

For $q^{-1}(x)$,
Domain: $x \geq 0$
Range: $q^{-1}(x) \leq 5$





To Draw the Inverse Graph from the Graph of an Original One-to-One Function:

Method 1: (Using $y = x$ line)
Draw the $y = x$ line. Reflect the Original Graph on that line.

Method 2: (Using Order Pairs)
Select a few important order pairs (like **x and y-intercepts**). **Switch the x and y values of each order pair** and re-graph.

Method 3: (Switch x and y axis)

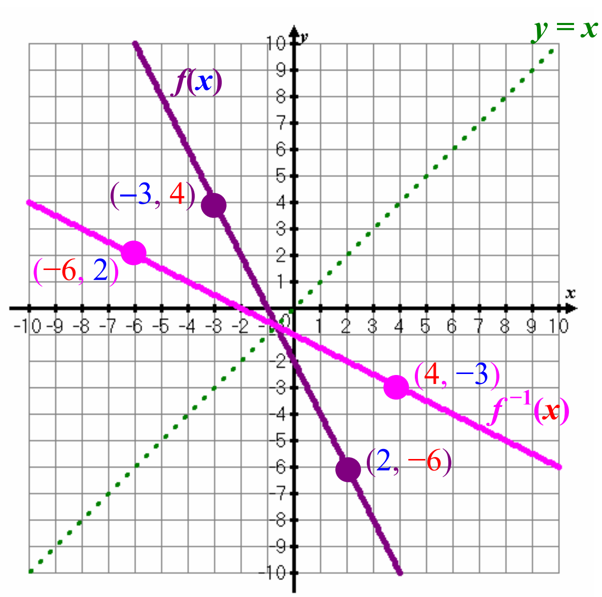
a. ***Redraw the graph*** **on a separate piece of paper.**

b. **Label "x" on the y-axis, and label the "y" on the x-axis.**

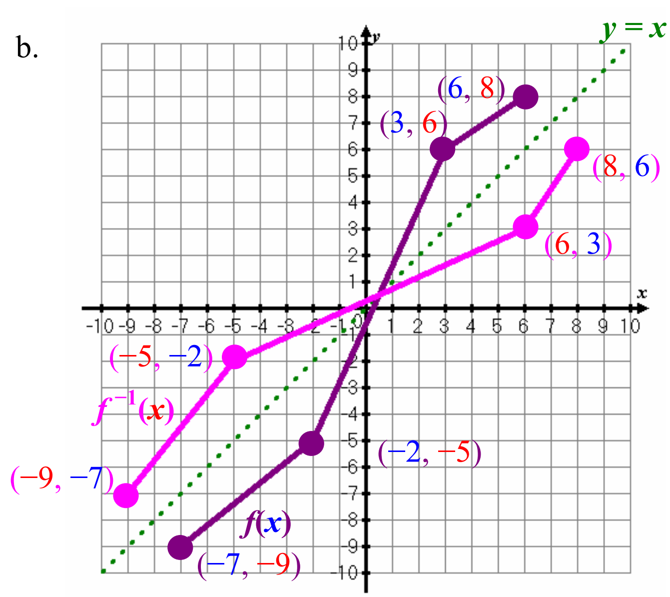
c. Look from the **backside of the paper**, and **rotate until the x-axis label is on the right hand side and the y-axis label is on top**. This is the shape and the orientation of the inverse graph.

Example 3: Sketch the inverse graph of the following functions.

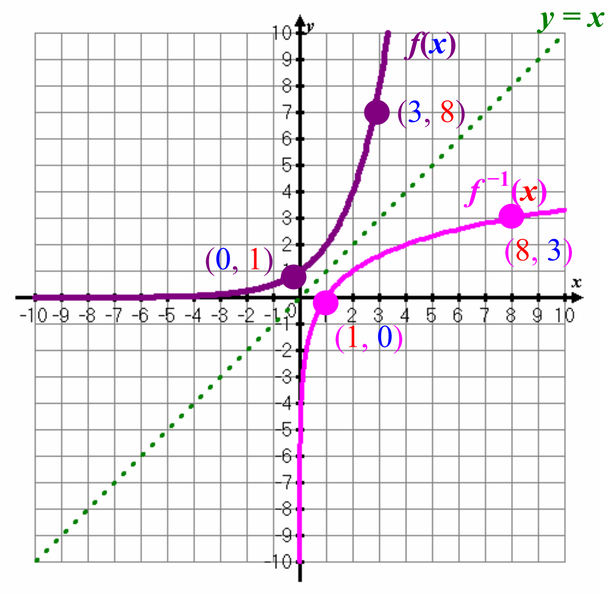
a.



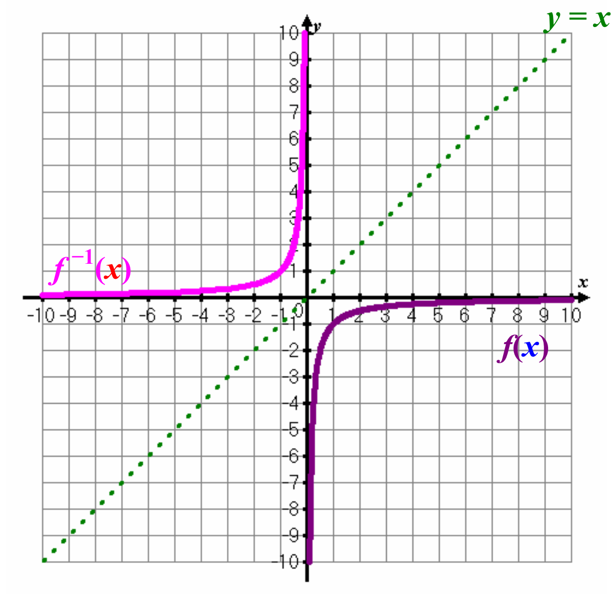
b.



c.



d.



Example 4: The temperature in degree Celsius is given by the function $C(F) = \frac{5}{9}(F - 32)$, where F is the temperature in Fahrenheit.

a. Find the inverse of the above function.
b. What does the inverse function represent?
c. Evaluate $C^{-1}(25)$. What does it mean?

a. $C(F) = \frac{5}{9}(F - 32)$

$C = \frac{5}{9}(F - 32)$ [use C instead of $C(F)$]

$F = \frac{5}{9}(C - 32)$ (switch C & F for inverse)

$\frac{9}{5}F = C - 32$ (Solve for the new C)

$\frac{9}{5}F + 32 = C$ $\quad$ $\mathbf{C^{-1}(F) = \frac{9}{5}F + 32}$

b. For $C(F)$, the input is temperature in Fahrenheit and the output is temperature in degree Celsius. **Hence $C^{-1}(F)$ means the output is temperature in Fahrenheit and the input is temperature in degree Celsius.**

c. Given 25°C, find Fahrenheit

$C^{-1}(25) = \frac{9}{5}(25) + 32$

$C^{-1}(25) = 45 + 32$

$\mathbf{C^{-1}(25) = 77}$ **Fahrenheit**

3-7 Assignment: pg. 279−281 #3, 5, 9, 11, 17, 19, 23, 33, 37, 47, 53, 69, 71, 75; Honours: #27, 39 and 79