# Chapter 1: Equations and Inequalities

## 1-1: Basic Equations

**Equations**: - two mathematical expressions separated by an equal signs where both expressions are equivalent.
- when **solving an equation**, we rearrange the equation by performing ***reverse operations*** in the ***reverse order of operation*** in order to isolate the desired variable, commonly known as **solution** or **root**.

**Equivalent Equations**: - when equations have the same solution.
- this usually happens when we multiply both sides of the equations by the same number or expression.

Examples: $x + 5 = 12$ is equivalent to $3(x + 5) = 3(12)$

$\frac{3}{2x+1} = -4$ is equivalent to $(2x + 1) \times \frac{3}{2x+1} = -4(2x + 1)$

**Linear Equations**: - equations where the variable is in of the first degree (exponent 1).

Examples: $5x - 2 = 4$ $\quad \frac{1}{2}y + 5 = -\frac{2}{3}y$ $\quad \sqrt{5}\,a - 4 = \frac{2a-8}{\sqrt{5}}$

**Nonlinear Equations**: - equations where the variable is **not** of the first degree.

Examples: $x^2 - 3x = 7$ (quadratic equation) $\quad \sqrt{x} - 5x = 2$ (radical equation)

$\frac{2}{x+1} - \frac{1}{x} = 3$ (rational equation)

**even index radical = ± solutions**
**odd index radical = one solution**

$5^x = 100$ (exponential equation) $\quad x^{3/4} - 5 = 12$ (fractional exponent equation)

**Extraneous Roots**: - solutions when substituted back into the original equation will render it undefined.

Examples: $\frac{2}{x+1} - \frac{1}{x} = 3$ Denominator cannot become 0. Hence, $x \neq 0$ or $-1$.

$\sqrt{x} - 5x = 2$ Radicand of an even index cannot be negative. Hence, $x \geq 0$.

**Example 1**: Classify the equations and find their solutions.

a. $\frac{3x}{2} - 5 = 4$

$\frac{3x}{2} = 4 + 5$

$3x = 9 \times 2$

$3x = 18$

$x = \frac{18}{3}$

$x = 6$

**Linear Equation ($x$ is of 1st degree)**

b. $\frac{5}{y+2} = \frac{2}{y-3}$

$\frac{5}{(y+2)} \times \frac{2}{(y-3)}$

$5(y - 3) = 2(y + 2)$

$5y - 15 = 2y + 4$

$5y - 2y = 4 + 15$

$3y = 19$

$y = \frac{19}{3}$

**You may only cross-multiply when there is a single fraction equal to another single fraction.**

**Nonlinear Equation ($y$ is in the denominator)**

**$y \neq -2$ or 3 and hence our solution is valid.**

c. $\frac{6}{3m-2} - 3 = \frac{3}{6m-4}$

$\frac{6}{(3m-2)} - 3 = \frac{3}{2(3m-2)}$

$\frac{6}{(3m-2)} - \frac{3}{2(3m-2)} = 3$

$\frac{6(2)-3}{2(3m-2)} \times \frac{3}{1}$

$9\,(1) = 3\,[2(3m-2)]$

$9 = 18m - 12$

$9 + 12 = 18m$

$21 = 18m$ $\quad m = \frac{7}{6}$

Must simplify to a single fraction equal to another single fraction before cross-multipling.

Nonlinear Equation ($y$ is in the denominator)

$m \neq \frac{2}{3}$ and $\therefore$ our solution is valid.

d. $4 - \frac{3}{x-2} = \frac{x-5}{x-2}$

$4 = \frac{x-5}{(x-2)} + \frac{3}{(x-2)}$

$\frac{4}{1} \times \frac{x-5+3}{(x-2)}$

$4(x-2) = 1(x-2)$

$4x - 8 = x - 2$

$4x - x = -2 + 8$

$3x = 6$

~~$x = 2$~~

Nonlinear Equation ($x$ is in the denominator)

$x \neq 2$ Hence our solution is extraneous.

e. $\sqrt{5}\,a - 4 = \frac{2a-8}{\sqrt{5}}$

$\frac{(\sqrt{5}a-4)}{1} \times \frac{(2a-8)}{\sqrt{5}}$

$\sqrt{5}\,(\sqrt{5}\,a - 4) = 1(2a-8)$

$5a - 4\sqrt{5} = 2a - 8$

$5a - 2a = -8 + 4\sqrt{5}$

$3a = -8 + 4\sqrt{5}$ $\quad a = \frac{-8+4\sqrt{5}}{3}$

Although there is a radical, it is a coefficient, the main variable is still in the 1st degree.

f. $(y+3)^2 = 6$

$(y+3) = \pm\sqrt{6}$

$y = -3 \pm \sqrt{6}$

Even index radical, we need to consider ± answers.

Nonlinear Equation ($y$ is in a bracket with an exponent right outside). In this case, it would have been a $y^2$.

g. $x^4 - 24 = 57$

$x^4 = 57 + 24$

$x^4 = 81$

$x = \pm\sqrt[4]{81}$

$x = \pm 3$

Nonlinear Equation ($x^4$ indicates the equation is in the 4th degree)

Even index radical, we need to consider ± answers.

h. $-6x^{2/3} + 15 = -9$

$-6x^{2/3} = -9 - 15$

$x^{2/3} = \frac{-24}{-6}$

$x^{2/3} = 4$

$\left(x^{2/3}\right)^{3/2} = (4)^{3/2} = \pm\sqrt[2]{(4)^3}$ $\quad x = \pm 8$

reverse operation of exponent $m/n$ is $n/m$

**Example 2**: Given the formulas below, solve for the variables indicated.

a. $V = \frac{4}{3}\pi r^3$ Solve for $r$

$\frac{3V}{4\pi} = r^3$

$\sqrt[3]{\frac{3V}{4\pi}} = \sqrt[3]{r^3}$

$\sqrt[3]{\frac{3V}{4\pi}} = r$

b. $SA = 2\pi r\,(r + H)$ Solve for $H$

$\frac{SA}{2\pi r} = (r + H)$

$\frac{SA}{2\pi r} - r = H$

**1-1 Assignment: pg. 78–79 #17, 21, 29, 33, 43, 51, 61, 69, 77, 83, 90, 93; Honours: #97**

## 1-2: Modeling Equations (Part 1)

**Modeling Equation**: - a process where we take a word problem and translate it into an algebraic equation.

**General Steps to Model Equations:**

1. ***Actively read the Question***: Ask what is required (unknown), what is given and try to understand the overall context of the question.
2. ***Define the Unknown***: Set a variable to represent the unknown.
3. ***Outline Given Information***: List the information given. If it can be explain by a picture, draw the diagram. (A picture is worth a thousand words!)
4. ***Set up the Model***: This can be a half-math, half-word equation, but it explains the relationship between the unknown variable and the known quantities. Translate it completely to an algebraic equation once you are satisfy with the model.
5. ***Solve the Equation***: Using reverse operations and reverse order of operations, we find the value of the unknown variable.
6. ***Check and Verify the Solution***: Take a look at the solution, does it make sense within the context of the question. Did you make any algebraic mistake? If the answer does not make sense, the original model was probably wrong, and need to be re-worked.

**Example 1**: Find three consecutive even integers whose sum is –288.

**Let $x$ = the smallest even integer**
$(x + 2)$ = the next larger even integer
$(x + 4)$ = the largest even integer

$$x + (x + 2) + (x + 4) = -288$$
$$3x + 6 = -288$$
$$3x = -288 - 6$$
$$3x = -294$$
$$x = \frac{-294}{3}$$

$x = -98$ $(x + 2) = -96$ $(x + 4) = -94$

**The even integers are –98, –96, –94**

**Example 2**: A salesperson received a 4% commission on her sales along with her base salary. In one month, she earned a total of $5200. If her annual base salary was $27,600, what was her sales amount for that month?

Let $x$ = sales amount

Earnings in a month = Monthly Base Salary + Commission
Earnings in a month = Annual Base Salary ÷ 12 + (Rate)(Sales Amount)

$$\$5200 = \frac{\$27{,}600}{12} + (0.04)x$$
$$\$5200 = \$2300 + 0.04x$$
$$\$5200 - \$2300 = 0.04x$$
$$\$2900 = 0.04x$$
$$\frac{\$2900}{0.04} = x$$

**$x$ = $72,500**

**Her Sales Amount was $72,500**

**Example 3**: A manufacturer has 500 L of a 5% vinegar solution. How much of a 40% vinegar solution can he add to bring the final concentration to 20%?

**Total Volume × Concentration of the Mixture = The Amount of Pure Vinegar**

| | Total Volume (L) | Concentration | Amount of Pure Vinegar (L) |
|---|---|---|---|
| **Initial Mixture** | 500 L | 0.05 | 25 |
| **Amount Added** | $x$ | 0.40 | $0.40x$ |
| **Final Mixture** | $(500 + x)$ | 0.20 | $0.20(500 + x)$ |

We can then add the amount of pure vinegar to form an equation.

$$25 + 0.40x = 0.20(500 + x)$$
$$25 + 0.40x = 100 + 0.20x$$
$$0.40x - 0.20x = 100 - 25$$
$$0.20x = 75$$

$x = 375$ L

**He must add 375 L of 40% vinegar solution to bring the final mixture to 20% concentration.**

**Example 4**: A small hose can fill an inflatable pool in 6 hours. A bigger hose can do the same job in two-third of the time. How long does it take to fill the pool if both hoses are used at the same time?

We ***cannot simply subtract the hours spent***. However, we can add their rates.

$$\text{Rate} = \frac{\text{Amount of Job Completed}}{\text{Time}}$$

Let $x$ = time takes to fill one pool if both hose working together

$$\frac{1\text{ pool}}{6\text{ hours}} + \frac{1\text{ pool}}{\tfrac{2}{3} \times 6\text{ hours}} = \frac{1\text{ pool}}{x\text{ hours}}$$

$$\frac{1}{6} + \frac{1}{4} = \frac{1}{x}$$

**(This is *NOT* $\frac{6}{1} + \frac{4}{1} = \frac{x}{1}$)**

$$\frac{5}{12} = \frac{1}{x}$$

$x = \frac{12}{5}$ **hours = 2.4 hours**

**Example 5**: John went for a 199 km mountain biking trip for two days. On day 1, he biked 3 km/h faster than day 2. If he biked for 9 hours on day 1 while on day 2 he biked for 11 hours, how fast was he traveling on each day?

**Use a *chart* to organize information for *distance, speed and time* word problems.**

**Distance = Speed × Time**

| | Distance (km) | Speed (km/h) | Time (h) |
|---|---|---|---|
| Day 1 | $9(x + 3)$ | $(x + 3)$ | 9 hours |
| Day 2 | $11x$ | $x$ | 11 hours |
| Total | 199 km | | |

$$199 = 9(x + 3) + 11x$$
$$199 = 9x + 27 + 11x$$
$$199 - 27 = 20x$$

$x = 8.6$ km/h  $(x + 3) = 11.6$ km/h

**On Day 1, John biked at 11.6 km/h. On Day 2, he biked at 8.6 km/h.**

**1-2 (Part 1) Assignment: pg. 89–92 #13, 17, 21, 23, 29, 35, 43, 49, 51; Honours: #55**

## 1-3: Quadratic Equations (Part 1)

**Quadratic Equation**

**An equation in a form of $ax^2 + bx + c = 0$ where $a \neq 0$**

**To Solve Quadratic Equations by Factoring:**

1. **Expand, Simplify, Cross-Multiply, and /or Rearrange the quadratic equation into the general form $ax^2 + bx + c = 0$ (with the preference for $a > 0$).**
2. **FACTOR the resulting general quadratic equation.**
3. **EQUATE each factor to 0 and solve for $x$.**

**$A \times B = 0$ means $A = 0$ or $B = 0$**

**Example 1**: Solve the following quadratic equations.

a. $x^2 + 8x + 15 = 0$

$(x+3)(x+5) = 0$

$(x+3) = 0$ or $(x+5) = 0$

**$x = -3$ and $x = -5$**

b. $2x^2 - 15 = -x$

$2x^2 + x - 15 = 0$

$(2x-5)(x+3) = 0$

$(2x-5) = 0$ or $(x+3) = 0$

**$x = \frac{5}{2}$ and $x = -3$**

c. $y^2 = 64$

$y^2 - 64 = 0$

$(y+8)(y-8) = 0$

$(y+8) = 0$ or $(y-8) = 0$

**$y = -8$ and $y = 8$**

d. $2w^2 = 8w$

$2w^2 - 8w = 0$

$2w(w-4) = 0$

$2w = 0$ or $(w-4) = 0$

**$w = 0$ and $w = 4$**

**The Quadratic Formula**

When finding solutions of a non-factorable quadratic equation algebraically, we can use the quadratic formula.

**To find EXACT SOLUTIONS of a NON-Factorable Quadratic Equation,**

$$ax^2 + bx + c = 0$$

**Quadratic Formula:** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ **(MEMORIZE!!)**

**Example 2**: Determine all real solutions of the equation.

a. $5x^2 + 1 = -7x$

$5x^2 + 7x + 1 = 0$

$a = 5 \quad b = 7 \quad c = 1$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(7) \pm \sqrt{(7)^2 - 4(5)(1)}}{2(5)}$

$x = \frac{-7 \pm \sqrt{29}}{10}$

$x = \frac{-7 + \sqrt{29}}{10}$ and $x = \frac{-7 - \sqrt{29}}{10}$

*Note that there are two solutions when the expression inside the square root of the quadratic formula yields a positive number.*

**Two solutions when $(b^2 - 4ac) > 0$.**

b. $x^2 - 6x = 2$

$x^2 - 6x - 2 = 0$

$a = 1 \quad b = -6 \quad c = -2$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-2)}}{2(1)}$

$x = \frac{6 \pm \sqrt{44}}{2} = \frac{6 \pm 2\sqrt{11}}{2}$

$x = 3 \pm \sqrt{11}$

***Common Mistakes***: $x(x - 6) = 2$

**Take out $x$ as a GCF instead of bringing all terms to one side and making the other side equal to 0.**

c. $3x^2 + 5 = 2x$

$3x^2 - 2x + 5 = 0$

$a = 3 \quad b = -2 \quad c = 5$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)}$

$x = \frac{2 \pm \sqrt{-56}}{6}$ **(No Solutions)**

**(cannot take a square root of a negative number)**

*Note that there are no solutions when the expression inside the square root of the quadratic formula yields a negative number.*

**No solutions when $(b^2 - 4ac) < 0$**

d. $4x^2 + 12x = -9$

$4x^2 + 12x + 9 = 0$

$a = 4 \quad b = 12 \quad c = 9$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(12) \pm \sqrt{(12)^2 - 4(4)(9)}}{2(4)}$

$x = \frac{-12 \pm \sqrt{0}}{8}$

$x = -\frac{3}{2}$

*Note that there is only one solutions when the expression inside the square root of the quadratic formula becomes 0.*

**One solution when $(b^2 - 4ac) = 0$**

**Discriminant**: - the part of the quadratic formula that determine the type of solution(s) of the equation.

**Quadratic Formula:** $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

**Discriminant = $b^2 - 4ac$**

**When Discriminant is *Positive*, $b^2 - 4ac > 0$ → *Two Distinct Real Roots***

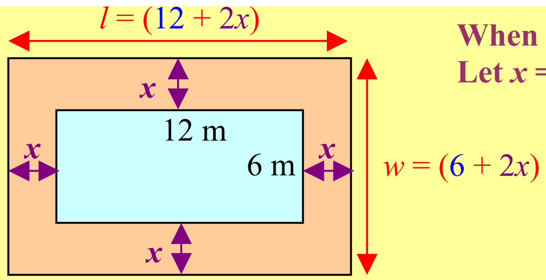
**When Discriminant is *Zero*, $b^2 - 4ac = 0$ → *One Distinct Real Root (or Two Equal Real Roots)***

**When Discriminant is *Negative*, $b^2 - 4ac < 0$ → *No Real Roots***

**1-3 (Part 1) Assignment: pg. 102– 103 #3, 7, 9, 29, 35, 41, 43, 47, 57, 59, 71, 77**

## 1-2 & 1-3: Modeling with Linear and Quadratic Equations (Part 2)

**Example 1**: A rectangular backyard swimming pool measures 6 m by 12 m. A tile border with uniform width is added around the pool. The perimeter of the border would be 56 m. How wide is the border surrounding the pool?




**When dealing with shapes and figure, *draw and label* a diagram.**
**Let $x$ = the width of the border**

$$P = 2l + 2w$$
$$56 = 2(12 + 2x) + 2(6 + 2x)$$
$$56 = 24 + 4x + 12 + 4x$$
$$56 = 36 + 8x$$
$$20 = 8x$$
$$\boldsymbol{x = 2.5 \text{ m}}$$

### Working Backwards to find Quadratic Equations

- when given two distinct or two equal rational solutions, we can form the factors of the quadratic equation.
- by multiplying the binomial factors afterwards, we can find the original quadratic equations.

**Example 2**: Given the roots below, write a quadratic equation.

a. 4, 4

$x = 4$ and $x = 4$ (Roots)
$(x - 4) = 0$ and $(x - 4) = 0$
$(x - 4)(x - 4) = 0$ (Factored Form)
$\boldsymbol{x^2 - 8x + 16 = 0}$ (Quadratic Form)

b. $0, \frac{1}{5}$

$x = 0$ and $x = \frac{1}{5}$
$x = 0$ and $5x = 1$
$(5x - 1) = 0$
$x(5x - 1) = 0$ $\boldsymbol{5x^2 - x = 0}$

c. $\frac{-3}{2}, \frac{5}{4}$

$x = -\frac{3}{2}$ and $x = \frac{5}{4}$
$2x = -3$ and $4x = 5$
$(2x + 3) = 0$ $(4x - 5) = 0$
$(2x + 3)(4x - 5) = 0$
$\boldsymbol{8x^2 + 2x - 15 = 0}$

### Problem Solving involving Quadratic Equations

- following the same rule as modeling linear equations with the exception that the equation will be in the second degree.
- move all terms to one side and let the other side equal to 0. Solve the equation either by factoring or using quadratic formula if it is not factorable.
- examine the solutions and eliminate any extraneous roots (for example: negative dimensions or negative time quantities).

**Example 3**: The sum of the squares from two consecutive even integers is 340. Find the two integers using an algebraic approach.

**Let $x$ = the smaller even integer**
$(x + 2)$ = the larger even integer

Sum of their squares means
$(x)^2 + (x + 2)^2$

$$(x)^2 + (x + 2)^2 = 340$$
$$x^2 + (x^2 + 4x + 4) = 340$$
$$2x^2 + 4x - 336 = 0$$
$$2(x^2 + 2x - 168) = 0$$
$$x^2 + 2x - 168 = \frac{0}{2}$$
$$(x - 12)(x + 14) = 0$$

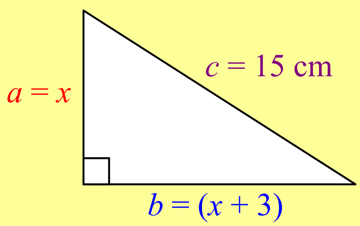
Case #1
$x - 12 = 0$
$\boldsymbol{x = 12}$
$\boldsymbol{(x + 2) = 14}$

Case #2
$x + 14 = 0$
$\boldsymbol{x = -14}$
$\boldsymbol{(x + 2) = -12}$

**The numbers are 12 and 14 or –14 and –12.**

**Example 4**: The hypotenuse of a right angle triangle measures 15 cm. Determine the lengths of the other two sides algebraically if they differ by 3 cm.




Applying the Pythagorean Theorem,

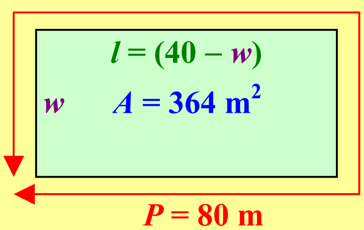
$$a^2 + b^2 = c^2$$
$$(x)^2 + (x + 3)^2 = (15)^2$$
$$x^2 + (x^2 + 6x + 9) = 225$$
$$2x^2 + 6x - 216 = 0$$
$$2(x^2 + 3x - 108) = 0$$

$$x^2 + 3x - 108 = \tfrac{0}{2}$$
$$(x - 9)(x + 12) = 0$$

$x = 9$ $\quad$ $x = -12$ (length can't be negative)

**The lengths of the two sides are 9 cm and 12 cm.**

**Example 5**: An 80 m fence is needed to surround a rectangular field of 364 $m^2$. Determine the dimensions of the field algebraically.




Let $w$ = width and $l$ = length

$$P = 2l + 2w$$
$$80 = 2l + 2w$$

(isolate $l$ to relate it with $w$ only)

$$80 - 2w = 2l$$
$$40 - w = l$$

$$A = lw$$
$$364 = (40 - w)w$$
$$364 = 40w - w^2$$
$$w^2 - 40w + 364 = 0$$
$$(w - 14)(w - 26) = 0$$

Case #1: $w = 14$, $l = 40 - 14 = 26$

Case #2: $w = 26$, $l = 40 - 26 = 14$

**The dimensions are 14 cm by 26 cm**

**Example 6**: An object was thrown upwards with an initial velocity of 19 m/s. Its height in metres and time spent in seconds can be expressed as $h = -5t^2 + 19t + 4$.

a. What is the height when the object was initially thrown?
b. Determine the time at which the object lands on the ground.

a. Object Initially Thrown at $t = 0$; $h = ?$

$$h = -5t^2 + 19t + 4$$
$$h = -5(0)^2 + 19(0) + 4$$

$\boldsymbol{h = 4}$ **m**

b. When object lands, $h = 0$; $t = ?$

$$h = -5t^2 + 19t + 4$$
$$0 = -5t^2 + 19t + 4$$
$$5t^2 - 19t - 4 = 0$$
$$(5t + 1)(t - 4) = 0$$

$t = -\tfrac{1}{5}$ $\quad$ $t = 4$

(time cannot be negative)

**It takes 4 seconds for the object to land.**

**Example 7**: Mary and Jane each left San Francisco and Gilroy respectively at the same time, and drove towards Santa Clara 150 km away. If Mary drove 10 km/h faster than Jane and she had to wait 10 minutes before Jane arrived at Santa Clara, how fast were both of them driving?

| | **Distance (km)** | **Speed (km/h)** | **Time (h)** | |
|---|---|---|---|---|
| Mary | 150 | $(x + 10)$ | $\dfrac{150}{(x+10)}$ | Less Time (speed is faster) |
| Jane | 150 | $x$ | $\dfrac{150}{x}$ | More Time (10 mins or $\tfrac{1}{6}$ hr slower) |

$$\frac{150}{x} - \frac{150}{(x+10)} = \frac{1}{6}$$

$x = 90$ **km/h**

**Mary = 100 km/h ; Jane = 90 km/h**

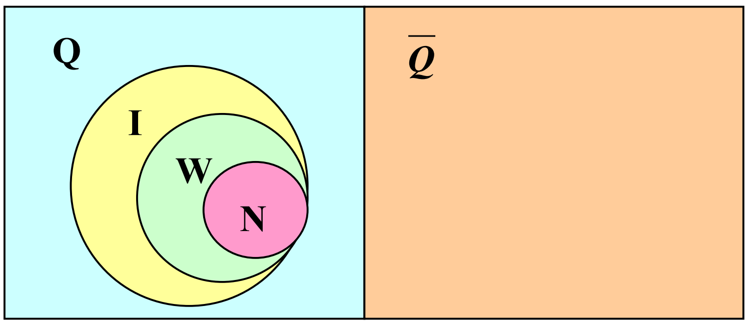
**1-2 & 1-3 (Part 2) Assignment: pg. 90–91 #33, 39, 45, 47;**
**pg. 103–105 #47, 57, 59, 71, 77, 81, 83, 89, 95; Honours: #102**

## 1-4: Complex Numbers

**Complex Numbers** ($\overline{R}$): - ***non-real numbers*** that cannot be represented on a number line or an $x$-$y$ plane.
- also called ***imaginary numbers***.

> **If $i^2 = -1$, then $i = \sqrt{-1}$ (imaginary unit)**
>
> ☛ ***Note:*** $\sqrt{-1} \neq -\sqrt{1}$ **(so $i \neq -1$)**

**Real Numbers ($R$)**




**Complex Numbers $(\overline{R})$**

Examples:

$$\sqrt{-40}$$

or

$$2\sqrt{-10} = 2\sqrt{-1} \times \sqrt{10}$$

or

$$2i\sqrt{10}$$

**Rational Numbers** ($Q$): - numbers that can be turned into a fraction $\frac{a}{b}$, where $a, b \in I$, and $b \neq 0$.
- include all Terminating or Repeating Decimals.
- include all Natural Numbers, Whole Numbers and Integers.
- include any perfect roots (radicals).

**Irrational Numbers** ($\overline{Q}$): - numbers that **CANNOT** be turned into a fraction $\frac{a}{b}$, where $a, b \in I$, and $b \neq 0$.
- include all non-terminating, non-repeating decimals.
- include any non-perfect roots (radicals).

**Real Numbers** ($R$): - any numbers that can be put on a number line.
- include all natural numbers, whole numbers, integers, rational and irrational numbers.

> **Principle Square Root of Negative Numbers**
>
> **For $-r < 0$, then $\sqrt{-r} = \sqrt{-1 \times r} = i\sqrt{r}$ (positive root only)**
>
> **Recall if $x^2 = a$, then $x = \pm\sqrt{a}$ . Similarly if $x^2 = -r$, then $x = \pm i\sqrt{r}$ (both positive & negative roots)**

**Example 1**: Simplify the following.

***Note***: $\left(\sqrt{-18}\right)\left(\sqrt{-18}\right) \neq \sqrt{(-18)(-18)}$

a. $\sqrt{-25}$

$= \sqrt{25 \times -1}$
$= \sqrt{25} \times \sqrt{-1}$
$= \mathbf{5i}$

b. $\sqrt{-50x^7y^4}$

$= \left(\sqrt{50 \times -1}\right)\left(x^3\sqrt{x}\right)\left(y^2\right)$
$= \left(\sqrt{25 \times 2} \times \sqrt{-1}\right)\left(x^3\sqrt{x}\right)\left(y^2\right)$
$= \mathbf{5ix^3y^2\sqrt{2x}}$

c. $\sqrt{(-18)^2}$

$= \sqrt{(-18)(-18)}$
$= \sqrt{324}$
$= \mathbf{18}$

**Follow order of operations!**

d. $\left(\sqrt{-18}\right)^2$

$= \left(\sqrt{-18}\right)\left(\sqrt{-18}\right)$
$= \left(i\sqrt{18}\right)\left(i\sqrt{18}\right)$
$= \left(i^2\right)\left(\sqrt{324}\right)$
$= (-1)(18)$
$= \mathbf{-18}$

**Example 2**: Find the value of $i$, $i^2$, $i^3$, …. $i^{10}$. Afterwards, determine the value for $i^{99}$.

$i = \sqrt{-1} = i$
$i^2 = -1$
$i^3 = (i^2)(i) = (-1)(i) = -i$
$i^4 = (i^2)(i^2) = (-1)(-1) = 1$
$i^5 = (i^4)(i) = (1)i = i$
$i^6 = (i^4)(i^2) = (1)(-1) = -1$
$i^7 = (i^4)(i^3) = (1)(-i) = -i$
$i^8 = (i^4)(i^4) = (1)(1) = 1$
$i^9 = (i^4)(i^4)(i) = (1)(1)i = i$
$i^{10} = (i^4)(i^4)(i^2) = (1)(1)(-1) = -1$

***Note the pattern:***

$i^1 = i$ $i^2 = -1$ $i^3 = -i$ $i^4 = 1$
$i^5 = i$ $i^6 = -1$ $i^7 = -i$ $i^8 = 1$
$i^9 = i$ $i^{10} = -1$ …..
**Pattern repeats every 4th power of *i*.**

$i^{99} = i^{4(24)+3}$
$= (i^4)^{24}(i^3)$
$= (1)^{24}(-i)$

$i^{99} = -i$

**Multiplying Complex Numbers**

$$\sqrt{-q} \times \sqrt{-r} = (i\sqrt{q})(i\sqrt{r}) = i^2\sqrt{qr}$$

☛ ***Note:*** $\sqrt{-q} \times \sqrt{-r} \neq \sqrt{(-q)(-r)}$ (so $\sqrt{-q} \times \sqrt{-r} \neq \sqrt{qr}$)

**Example 3**: Simplify the following.

a. $(\sqrt{-6})(\sqrt{3})$

*Note*: $(\sqrt{-6})(\sqrt{3}) \neq (-\sqrt{6})(\sqrt{3})$

$= (i\sqrt{6})(\sqrt{3})$
$= i\sqrt{18}$
$= i\sqrt{9 \times 2} = 3i\sqrt{2}$

b. $(\sqrt{-8})(\sqrt{-2})$

*Note*: $(\sqrt{-8})(\sqrt{-2}) \neq \sqrt{(-8)(-2)}$

$= (i\sqrt{8})(i\sqrt{2})$
$= i^2\sqrt{16}$
$= (-1)(4) = -4$

c. $(7i)(4i)$

$= 28i^2$
$= 28(-1) = -28$

d. $(6i)^2$

***Note:*** $(6i)^2 \neq 6i^2$

$= (6i)(6i)$
$= 36i^2$
$= 36(-1) = -36$

e. $-(7i)^2$

f. $(-7i)^2$

The exponent only "sees" the bracket or the variable immediately preceding it.

e.
$= -(7i)(7i)$
$= -(49i^2)$
$= -(49 \times -1) = 49$

f.
$= (-7i)(-7i)$
$= 49i^2$
$= 49(-1) = -49$

**Composite Complex Number** ($a + bi$)
**($a$ is real number and $bi$ is the imaginary number)**

**Operations with Composite Complex Numbers**

- ***add and subtract like terms*** (treat $i$ terms as like-terms).
- ***multiply*** composite complex numbers like expanding binomials by doing **FOIL**. ***Combine like-terms*** and ***simplify***. **(Remember, $i^2 = -1$)**

**Example 4**: Simplify the following.

a. $(4 - 5i) + (-6 - 2i)$

$= 4 - 5i - 6 - 2i$
$= -2 - 7i$

***Adding*: drop brackets and combine like terms**

b. $(7 - 8i) - (9 - 3i)$

$= 7 - 8i - 9 + 3i$
$= -2 - 5i$

***Subtracting*: drop brackets, switch signs in bracket after the − sign combine like terms**

c. $(2 - 3i)(1 + i)$

$= (2 - 3i)(1 + i)$
$= 2 + 2i - 3i - 3i^2$
$= 2 - i - 3(-1)$
$= 2 - i + 3 \quad = \boxed{5 - i}$

***Multiplying*: FOIL, combine like terms and simplify.**

d. $(5 - 7i)^2$

$= (5 - 7i)(5 - 7i)$
$= 25 - 35i - 35i + 49i^2$
$= 25 - 70i + 49(-1)$
$= 25 - 70i - 49 \quad = \boxed{-24 - 70i}$

***Note*: $(5 - 7i)^2 \neq (25 + 49i^2)$**

**Conjugate of Complex Number**

**$(a + bi)$ is a conjugate of $(a - bi)$**

**Product of Conjugate Complex Numbers**

$(a + bi)(a - bi) = a^2 - b^2i^2 = a^2 - b^2(-1)$
$(a + bi)(a - bi) = a^2 + b^2$

**Example 5**: Simplify the following.

a. $(4 - 3i)(4 + 3i)$

$= (4 - 3i)(4 + 3i)$
$= 16 + 12i - 12i - 9i^2$
$= 16 - 9(-1)$
$= 16 + 9 \quad = \boxed{25}$

b. $(7 + 6i)(7 - 6i)$

$= (7 + 6i)(7 - 6i)$
$= 49 - 36i^2$
$= 49 - 36(-1)$
$= 49 + 36 \quad = \boxed{85}$

c. $(3 - 8i)^2(3 + 8i)^2$

$= (3 - 8i)(3 - 8i)\,(3 + 8i)(3 + 8i)$ — *conjugates* / *conjugates*
$= [(3 - 8i)(3 + 8i)][(3 - 8i)(3 + 8i)]$
$= [9 - 64i^2][9 - 64i^2]$
$= [9 - 64(-1)][9 - 64(-1)]$
$= (73)(73) \quad = \boxed{5329}$

***Note that when multiplying composite complex number conjugates, the middle two terms cancel out, leaving a real number.***

**Rationalizing Complex Numbers**: - similar to rationalizing radical binomial expression, we ***multiply a by a fraction consist of the conjugate of the denominator over itself***.

**Example 6**: Simplify the following.

a. $\dfrac{8}{\sqrt{-4}}$

$= \dfrac{8}{i\sqrt{4}} = \dfrac{8}{2i} = \dfrac{4}{i}$

$= \dfrac{4}{i} \times \dfrac{i}{i}$ **Rationalize Denominator**

$= \dfrac{4i}{i^2} = \dfrac{4i}{(-1)} \quad = \boxed{-4i}$

b. $\dfrac{6}{3 - 5i}$

**Multiply Conjugate over itself for Rationalization**

$= \dfrac{6}{(3 - 5i)} \times \dfrac{(3 + 5i)}{(3 + 5i)}$

$= \dfrac{6(3 + 5i)}{9 - 25i^2} = \dfrac{6(3 + 5i)}{9 - 25(-1)}$

$= \dfrac{6(3 + 5i)}{34} \quad = \boxed{\dfrac{3(3 + 5i)}{17}}$

c. $\dfrac{4 + i}{2 - 3i}$

$= \dfrac{(4 + i)}{(2 - 3i)} \times \dfrac{(2 + 3i)}{(2 + 3i)}$

$= \dfrac{(4 + i)(2 + 3i)}{4 - 9i^2}$

$= \dfrac{8 + 12i + 2i + 3i^2}{4 - 9(-1)}$

$= \dfrac{8 + 14i + 3(-1)}{13} \quad = \boxed{\dfrac{5 + 14i}{13}}$

**Example 7**: Find all solutions of the equation, $2x^2 - 7x + 8 = 0$ and express them in the form of $a + bi$.

$2x^2 - 7x + 8 = 0$

$a = 2 \quad b = -7 \quad c = 8$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(8)}}{2(2)}$

$x = \dfrac{7 \pm \sqrt{-15}}{4}$

$\boxed{x = \dfrac{7 \pm i\sqrt{15}}{4}}$

**(No Real Solutions, but we have complex solutions.)**

**1-4 Assignment: pg. 110–111 #13, 19, 23, 25, 31, 37, 41, 42, 45, 49, 55, 59; Honours: #75**

## 1-5: Other Types of Equations

**Solving (Non-Linear) Polynomial Equations: ($ax^n + bx^{n-1} + cx^{n-2} + \ldots + dx$ + constant = 0 for $n \in N$)**

1. ***Bring all terms to one side of the equation and let the other side equals to zero***. Combine any like terms. (Do **NOT CANCEL** any $x$ as monomial by dividing $x^n$ on both sides).
2. ***Factor by common factor, grouping, or substitution***.
3. ***Equate each*** linear ***factor to zero*** and solve for $x$. Equate each quadratic factor to zero. Either solve for $x$ by factoring further or use quadratic formula. Monomial factors have solutions of 0.

**Example 1**: Solve the following equations.

a. $8x^2 = 12x^5$

$0 = 12x^5 - 8x^2$ **Factor by GCF**

$0 = 4x^2(3x^3 - 2)$

$4x^2 = 0$ $\quad$ $3x^3 - 2 = 0$

$\boldsymbol{x = 0}$ $\quad$ $x^3 = \frac{2}{3}$

$\boldsymbol{x = \sqrt[3]{\frac{2}{3}}}$

**WRONG WAY!**

$\frac{8x^2}{4x^2} = \frac{12x^5}{4x^2}$

$2 = 3x^5$

$\frac{2}{3} = x^3 \quad x = \sqrt[3]{\frac{2}{3}}$

**Note that one of the solutions is eliminated!**

b. $3x^3 + 18x = 2x^2 + 12$

**Factor by Grouping**

$3x^3 - 2x^2 + 18x - 12 = 0$

$(3x^3 - 2x^2) + (18x - 12) = 0$

$x^2(3x - 2) + 6(3x - 2) = 0$

$(3x - 2)(x^2 - 6) = 0$

$3x - 2 = 0$ $\quad$ $x^2 - 6 = 0$

$x^2 = 6$

$\boldsymbol{x = \frac{2}{3}}$ $\quad$ $\boldsymbol{x = \pm\sqrt{6}}$

**Solving Rational Equations: (Equations with variable terms in denominators)**

1. ***Bring ALL* Fractional Terms containing the Variable *to one side of the equation***. Leave all whole number or whole number fractions at the other side of the equation.
2. **Combine *the rational expressions at one side using* Common Denominator & Equivalent Fractions**.
3. **Cross Multiply and Solve**.
4. If the resulting equation turns out to be a quadratic equation, there could be two solutions. Remember to **VERIFY *all solutions and be mindful about* Extraneous Roots**.

**Example 2**: Solve $-\frac{12}{x} = \frac{4}{x-2} - 5$.

**Restrictions** (denominator cannot be zero)

$\boldsymbol{x \neq 0}$ $\quad$ $x - 2 \neq 0$

$\boldsymbol{x \neq 2}$

$-\frac{12}{x} - \frac{4}{x-2} = -5$

$+\frac{12}{x} + \frac{4}{x-2} = +5$ (Multiply Both Sides by −1)

$\frac{12(x-2) + 4(x)}{x(x-2)} = 5$ **(Common Denominator and Equivalent Fractions)**

$\frac{12x - 24 + 4x}{x^2 - 2x} = \frac{5}{1}$ **(Cross Multiply)**

$16x - 24 = 5(x^2 - 2x)$

$16x - 24 = 5x^2 - 10x$

$0 = 5x^2 - 10x - 16x + 24$

$0 = 5x^2 - 26x + 24$ **(Quadratic Equation)**

$0 = (5x - 6)(x - 4)$ **(Factor)**

$0 = (5x - 6)(x - 4)$

$5x - 6 = 0$ $\quad$ $x - 4 = 0$

$5x = 6$

$\boldsymbol{x = \frac{6}{5}}$ $\quad$ $\boldsymbol{x = 4}$

**Actual solutions are not the restrictions. Hence, they are both valid.**

**Solving Radical Equations: (Equations with expressions in Square Roots)**

1. For Equations with **ONE Radical sign**:
   a. ***Isolate the Radical Expression on one side of the equation*.**
   b. ***Square* BOTH Sides.**
   c. ***Expand*, *Simplify* and *Solve*.**
2. For Equations with **TWO Radical signs with NO Other Terms**:
   a. ***Equate the two Radical Expressions on EITHER side of the equation*.**
   b. ***Square* BOTH Sides. (Not just cancelling out the two radical signs!)**
   c. ***Expand*, *Simplify* and *Solve*.**
3. **VERIFY *the solutions by Substitution.* Eliminate any Extraneous Roots.**

**Example 3**: Solve the following equations.

a. $2x = 7 - \sqrt{4-x}$

**Restriction** (radicand has to be greater than or equal to zero)

$4 - x \geq 0$
$-x \geq -4$
$\boldsymbol{x \leq 4}$

(Multiply or Divide both Side by negative. We have to switch direction of the inequality sign.)

$2x = 7 - \sqrt{4-x}$

$\sqrt{4-x} = 7 - 2x$ **(Isolate the single radical term on one side)**

$\left(\sqrt{4-x}\right)^2 = (7-2x)^2$ **(Square Both Sides)**

$4 - x = (7-2x)(7-2x)$

$4 - x = 49 - 14x - 14x + 4x^2$

$0 = 4x^2 - 27x + 45$ **(Quadratic Equation)**

$0 = (4x - 15)(x - 3)$ **(Factor)**

$0 = (4x - 15)(x - 3)$

$4x - 15 = 0 \qquad x - 3 = 0$

$4x = 15$

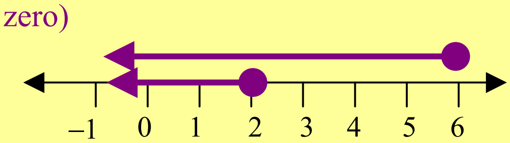
$\boldsymbol{x = \frac{15}{4}} \qquad \boldsymbol{x = 3}$

**Actual solutions are not the restriction. Hence, they are both valid.**

b. $\sqrt{6-x} - \sqrt{4-2x} = 0$

**Restriction** (radicand has to be greater than or equal to zero)

$6 - x \geq 0 \qquad 4 - 2x \geq 0$
$-x \geq -6 \qquad -2x \geq -4$
$\boldsymbol{x \leq 6}$ **and** $\boldsymbol{x \leq 2}$




$\sqrt{6-x} - \sqrt{4-2x} = 0$

$\sqrt{6-x} = \sqrt{4-2x}$ **(Isolate each radical term on either side)**

$\left(\sqrt{6-x}\right)^2 = \left(\sqrt{4-2x}\right)^2$ **(Square Both Sides)**

$6 - x = 4 - 2x$

$-x + 2x = 4 - 6$ **(Linear Equation)**

$\boldsymbol{x = -2}$ **Actual solution is not the restriction. Hence, it is valid.**

**Equations with Recurring Expressions and Quadratic Type:**

1. For equations with ***Recurring Expressions*** **(brackets with the same expression)**:
   a. **SUBSTITUTE a *Single Variable* in its place**. State this substitution.
   b. ***Solve*** this simplified equation. (It would most likely be a quadratic equation).
   c. **Equate *this solution to the Recurring Expression*** and **Solve *for the Original Variable***.

2. For equations in the forms of $ax^4 + bx^2 + c = 0$ and $ax^6 + bx^3 + c = 0$:
   a. **Treat *and* Factor *them like the* Regular QUADRATIC form**. However, the factored form would be $(dx^2 + e)(px^2 + q) = 0$ and $(dx^3 + e)(px^3 + q) = 0$ respectively.
   b. **Equate *each factor to Zero*** and **Solve**.

**Example 4**: Solve the following equations.

a. $(x - 3)^2 - 8(x - 3) + 12 = 0$

Recurring Bracket = $(x - 3)$ **Let $A = (x - 3)$**

$$(x - 3)^2 - 8(x - 3) + 12 = 0$$
$$A^2 - 8A + 12 = 0$$
$$(A - 6)(A - 2) = 0$$

$A - 6 = 0$ $A - 2 = 0$

$A = 6$ $A = 2$

**Substitute $(x - 3)$ back into $A$**

$x - 3 = 6$ $x - 3 = 2$

$\boldsymbol{x = 9}$ $\boldsymbol{x = 5}$

b. $x^4 - 8x^2 = 48$

$x^4 - 8x^2 - 48 = 0$ (Quadratic Type)

$(x^2 - 12)(x^2 + 4) = 0$

$x^2 - 12 = 0$ $x^2 + 4 = 0$

$x^2 = 12$ $x^2 = -4$

$x = \pm\sqrt{12}$ $x = \pm\sqrt{-4}$

$\boldsymbol{x = \pm 2\sqrt{3}}$ $\boldsymbol{x = \pm 2i}$

**All solutions include Real & Complex Solutions.**

**Example 5**: A children birthday party has a fixed cost of \$400. At the last minute, four more children showed up without RSVP (prior reservation). This lowers the cost per child by \$5. How many children's parents bothered to RSVP to this party?

| | **RSVPs** | **RSVPs and Party Crashers** |
|---|---|---|
| **Total Cost** | \$400 | \$400 |
| **Number of Children** | $x$ | $(x + 4)$ |
| **Cost per Person** | $\frac{\$400}{x}$ (Cost More per Person) | $\frac{\$400}{(x+4)}$ (Cost Less per Person) |

$$\frac{400}{x} - \frac{400}{(x+4)} = 5$$
$$\frac{400(x+4) - 400(x)}{x(x+4)} = 5$$
$$\frac{400x + 1600 - 400x}{x(x+4)} = \frac{5}{1} \quad \textbf{(Cross Multiply)}$$
$$1600 = 5x(x + 4)$$
$$0 = 5x^2 + 20x - 1600$$

$$0 = 5x^2 + 20x - 1600$$
$$0 = 5(x^2 + 16x - 320)$$
$$0 = 5(x - 16)(x + 20)$$

$x - 16 = 0$ $x + 20 = 0$

$\boldsymbol{x = 16}$ **children** ~~$x = -20$~~

(cannot have negative number of children)

**1-5 Assignment: pg. 118–119 #5, 11, 15, 19, 23, 27, 33, 41, 45, 59, 69, 71;**
**Honours: #53, 74**

## 1-6: Inequalities

**Linear Inequalities**: - inequalities where the highest degree of the variable term is 1 and there are no variables in the denominator, radical sign, or with fractional exponent.

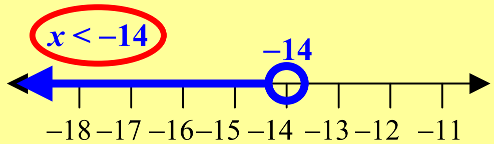
**Solving Linear Inequalities**:

1. ***Solve*** **the inequalities** ***like they were equations***. ***Treat the inequality sign like it was an equal sign***.
2. **Always** ***ISOLATE*** **the** ***Variable*** **on the** ***LEFT side of the inequality***.
3. **When** ***Dividing or Multiplying by a negative number on BOTH sides of the inequality*** **(bringing a negative number from one side to the other to Multiply or Divide),** ***SWITCH the direction of the inequality sign***.

**Example 1**: Solve the following inequalities. Graph the solution.

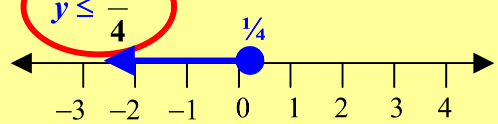
a. $-5x - 7 > 63$

$-5x - 7 > 63$
$-5x > 63 + 7$
$-5x > 70$
$x < \frac{70}{-5}$ **Switch Inequality Sign (Divided by a Negative)**
$x < -14$




b. $3(2y + 4) \le -2(5y - 8)$

$3(2y + 4) \le -2(5y - 8)$
$6y + 12 \le -10y + 16$
$6y + 10y \le 16 - 12$
$16y \le 4$
$y \le \frac{4}{16}$ Do NOT Switch Inequality Sign (did not divide by a Negative)
$y \le \frac{1}{4}$



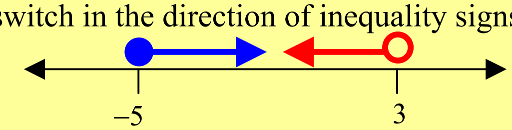

**Simultaneous Inequalities**: - when there are two inequality symbols surrounding an algebraic expression.

**Solving Simultaneous Inequalities**:

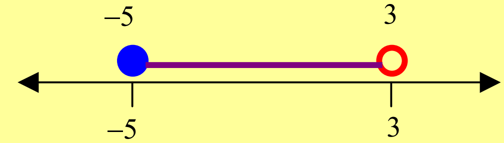
1. ***Work with both side of the inequalities at the same time***. **Follow the rules of Reverse Operation and Reverse Order of Operation.**
2. **Remember to** ***SWITCH the direction of the inequality sign***, **when** ***Dividing or Multiplying by a negative number on BOTH sides of the inequality***.

**Example 2**: Solve the $-4 < 5 - 3x \le 20$. Graph the solution. Express it as inequality as well as interval notation.

$-4 < 5 - 3x \le 20$
$-4 - 5 < -3x \le 20 - 5$ (Add 5 to either sides)
$\frac{-9}{-3} > x \ge \frac{15}{-3}$ (Divide both sides by −3. Note the switch in the direction of inequality signs.)
$3 > x \ge -5$




$-5 \le x < 3$
$[-5, 3)$ (Rearrange Inequality Statement)



**Nonlinear Inequalities**: - inequalities where the highest degree of the variable term is not 1 (polynomial inequality) or there are variables in the denominator (rational inequality), radical sign, or with fractional exponent.

**General Rules to Solve Nonlinear Inequalities: (*This includes Polynomial and Rational Inequalities*)**

1. ***Bring ALL Terms to the LEFT side of the inequality*** **and** ***Let the RIGHT side become Zero***.
   - **a.** If it is a ***POLYNOMIAL*** **inequality**, **FACTOR** it.
   - **b.** If it is a ***RATIONAL*** **inequality**, find ***Common denominator***, ***Equivalent fractions*** and ***Simplify***.
   - - ***We solve them differently than linear inequalities because there are only a limited number of ways to combine factors or quotients such that the result is either positive ( > 0 ) or negative ( < 0 ).***
2. ***EQUATE each Factor and/or Denominator/Numerator linear expression to ZERO and SOLVE***. These are the **test-values**.
3. **Using a Number Line, set up a CHART using these *test-values* as BOUNDARIES.** ***Select values from each region to test the (+ or −) cases*.** **Include the cases that agree with the initial inequality as the final solution**.

**Signs of Products and Quotients**

- **POSITIVE Product or Quotient** **when it has** **EVEN Number of *Negative Cases.***
- **NEGATIVE Product or Quotient** **when it has** **ODD Number of *Negative Cases.***

**Example 3**: Solve the following inequalities.

a. $x^2 - 3x < 10$

$x^2 - 3x - 10 < 0$ (Rearrange one side to zero)

$(x - 5)(x + 2) < 0$ (Factor)

$x - 5 = 0$ $\quad$ $x + 2 = 0$

$\boldsymbol{x = 5}$ $\quad$ $\boldsymbol{x = -2}$

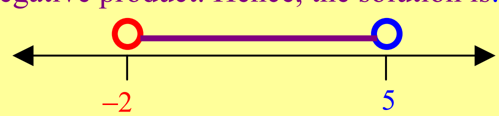
**Test-Values (Boundaries on Number Line)**

$x < -2$ | $-2$ | $\boldsymbol{-2 < x < 5}$ | $5$ | $x > 5$

| Regions | $(x - 5)$ | $(x + 2)$ | $(x - 5)(x + 2)$ |
|---|---|---|---|
| $x < -2$ | − | − | + |
| $-2 < x < 5$ | − | + | − |
| $x > 5$ | + | + | + |

For $(x - 5)(x + 2) < 0$ (negative), the region $-2 < x < 5$ yields a negative product. Hence, the solution is:

**Inequality Form: $-2 < x < 5$** $\quad$ **Interval Form: $(-2, 5)$**




b. $(x - 1)(x + 2)(x - 6) \geq 0$

$(x - 1)(x + 2)(x - 6) \geq 0$

$x - 1 = 0$ $\quad$ $x + 2 = 0$ $\quad$ $x - 6 = 0$

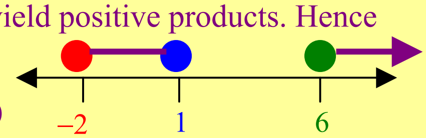
$\boldsymbol{x = 1}$ $\quad$ $\boldsymbol{x = -2}$ $\quad$ $\boldsymbol{x = 6}$

**Test-Values**

$x \leq -2$ | $-2$ | $\boldsymbol{-2 \leq x \leq 1}$ | $1$ | $1 \leq x \leq 6$ | $6$ | $\boldsymbol{x \geq 6}$

| Regions | $(x - 1)$ | $(x + 2)$ | $(x - 6)$ | $(x - 1)(x + 2)(x - 6)$ |
|---|---|---|---|---|
| $x \leq -2$ | − | − | − | − |
| $-2 \leq x \leq 1$ | − | + | − | + |
| $1 \leq x \leq 6$ | + | + | − | − |
| $x \geq 6$ | + | + | + | + |

For $(x - 1)(x + 2)(x - 6) \geq 0$ (positive), the regions $-2 \leq x \leq 1$ with $x \geq 6$ yield positive products. Hence the solution is:




**Inequality Form: $-2 \leq x \leq 1$ or $x \geq 6$** $\quad$ **Interval Form: $[-2, 5] \cup [6, \infty)$**

c. $\frac{k+4}{k-2} \le 3$

**CANNOT Cross-Multiply** because we do not know which inequality is valid.

$$\frac{k+4}{k-2} \le \frac{3}{1} \qquad 1(k+4) \le 3(k-2) \quad \textbf{OR ? ? ?} \quad 3(k-2) \le 1(k+4)$$

$\frac{(k+4)}{(k-2)} - \frac{3}{1} \le 0$ (Bring all terms to one side)

$\frac{(k+4)(1)-3(k-2)}{(k-2)} \le 0$ **(Common Denominator & Equivalent Fractions)**

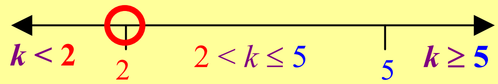
$\frac{k+4-3k+6}{(k-2)} \le 0$

$\frac{-2k+10}{(k-2)} \le 0$

$-2k+10=0$ $\quad k-2=0$ **Restriction:** $k-2 \ne 0$

$\boldsymbol{k=5}$ $\quad \boldsymbol{k=2}$ $\quad \boldsymbol{k \ne 2}$

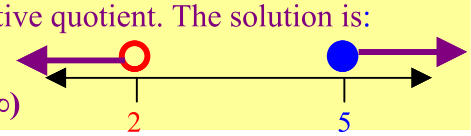
**Test-Values**




| Regions | $(-2k+10)$ | $(k-2)$ | $\frac{-2k+10}{(k-2)}$ |
|---|---|---|---|
| $k<2$ | + | – | – |
| $2<k\le 5$ | + | + | + |
| $k\ge 5$ | – | + | – |

**Since $k \ne 2$, we cannot say $k \le 2$ or $2 \le k \le 5$, but $k < 2$ or $2 < k \le 5$**

For $\frac{-2k+10}{(k-2)} \le 0$ (negative), the regions $k < 2$ with $k \ge 5$ yield a negative quotient. The solution is:




**Inequality Form: $k < 2$ or $k \ge 5$** **Interval Form: $(-\infty, 2) \cup [5, \infty)$**

d. $\frac{3}{2t-1} \le \frac{2}{t+1}$

**CANNOT Cross-Multiply** because we do not know which inequality is valid.

$$\frac{3}{2t-1} \le \frac{2}{t+1} \qquad 3(t+1) \le 2(2t-1) \quad \textbf{OR ? ? ?} \quad 2(2t-1) \le 3(t+1)$$

$\frac{3}{2t-1} - \frac{2}{t+1} \le 0$ (Bring all terms to one side)

$\frac{3(t+1)-2(2t-1)}{(2t-1)(t+1)} \le 0$ **(Common Denominator & Equivalent Fractions)**

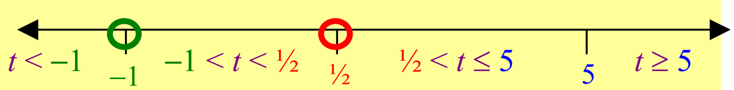
$\frac{3t+3-4t+2}{(2t-1)(t+1)} \le 0$ **Restriction:**

$\frac{-t+5}{(2t-1)(t+1)} \le 0$ $\quad 2t-1 \ne 0 \quad t+1 \ne 0$

$\boldsymbol{t \ne ½} \quad \boldsymbol{t \ne -1}$

$-t+5=0 \quad 2t-1=0 \quad t+1=0$

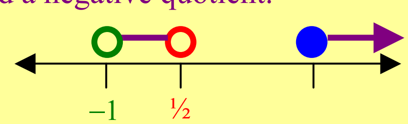
$\boldsymbol{t=5} \quad \boldsymbol{t=½} \quad \boldsymbol{t=-1}$




| Regions | $(-t+5)$ | $(2t-1)$ | $(t+1)$ | $\frac{-t+5}{(2t-1)(t+1)}$ |
|---|---|---|---|---|
| $t<-1$ | + | – | – | + |
| $-1<t<½$ | + | – | + | – |
| $½<t\le 5$ | + | + | + | + |
| $t\ge 5$ | – | + | + | – |

**Since $t \ne ½$ or $t \ne -1$, we cannot say $t \le -1$ or $-1 \le t \le ½$ or $½ \le t \le 5$, but $t < -1$ or $-1 < t < ½$ or $½ < t \le 5$**

For $\frac{-t+5}{(2t-1)(t+1)} \le 0$ (negative), the regions $-1 < t < ½$ with $t \ge 5$ yield a negative quotient.




**Inequality Form: $-1 < t < ½$ or $t \ge 5$**
**Interval Form: $(-1, ½) \cup [5, \infty)$**

e. $x - 1 < \dfrac{6}{x-2}$

$\dfrac{x}{1} - \dfrac{1}{1} - \dfrac{6}{x-2} < 0$ (Bring all terms to one side)

$\dfrac{x(x-2) - 1(x-2) - 6}{(x-2)} < 0$ **(Common Denominator & Equivalent Fractions)**

$\dfrac{x^2 - 2x - x + 2 - 6}{(x-2)} < 0$ **Restriction:** $x - 2 \neq 0$, $\boldsymbol{x \neq 2}$

$\dfrac{x^2 - 3x - 4}{(x-2)} < 0$ **Factor Numerator**

$\dfrac{(x-4)(x+1)}{(x-2)} < 0$

**CANNOT Cross-Multiply** because we do not know which inequality is valid.

$\dfrac{x-1}{1} < \dfrac{6}{x-2}$

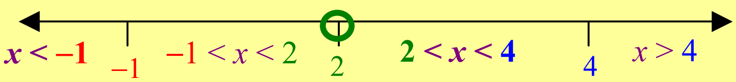
$(x - 1)(x - 2) < 1(6)$

**OR ? ? ?**

$1(6) < (x - 1)(x - 2)$

$x - 4 = 0$, $\boldsymbol{x = 4}$ $x + 1 = 0$, $\boldsymbol{x = -1}$ $x - 2 = 0$, $\boldsymbol{x = 2}$

**Test Values**

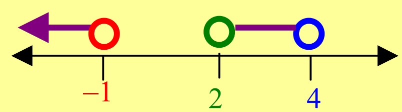



| **Regions** | $(x - 4)$ | $(x + 1)$ | $(x - 2)$ | $\dfrac{(x-4)(x+1)}{(x-2)}$ |
|---|---|---|---|---|
| $x < -1$ | – | – | – | – |
| $-1 < x < 2$ | – | + | – | + |
| $2 < x < 4$ | – | + | + | – |
| $x > 4$ | + | + | + | + |

For $\dfrac{(x-4)(x+1)}{(x-2)} < 0$ (negative), the regions $x < -1$ with $2 < x < 4$ yield a negative quotient.

**Inequality Form:** $\boldsymbol{x < -1}$ **or** $\boldsymbol{2 < x < 4}$

**Interval Form:** $\boldsymbol{(-\infty, -1) \cup (2, 4)}$




**Example 4**: *ABC* cellular charges \$30/month for the first 400 minutes and \$0.60/minute after. *XYZ* cellular charges \$50/month for unlimited calling. For how many minutes of talk time does *XYZ* cellular become more of a bargain than *ABC* cellular?

*ABC* Cost = \$30/month + \$0.60/min after first 400 minutes *XYZ* Cost = \$50/month (unlimited time)

Let $t$ = monthly talk time in minutes

***ABC* Cost = 30 + 0.60($t$ – 400)** ***XYZ* Cost = 50**

**When *XYZ* Cost < *ABC* Cost,**

$$50 < 30 + 0.60(t - 400)$$
$$50 < 30 + 0.60t - 240$$
$$50 < 0.60t - 210$$
$$50 + 210 < 0.60t$$
$$\frac{260}{0.60} < t$$
$$433.33 < t$$

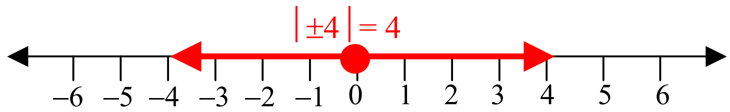
$\boldsymbol{t > 433}$ **min**

**It is cheaper to go with *XYZ* Cellular when the monthly talktime exceeds 433 minutes.**

**1-6 Assignment: pg. 128–130 #13, 21, 25, 31, 35, 39, 43, 48, 53, 61, 67, 75, 83; Honours: #86**

## 1-7: Absolute Value Equation and Inequalities

**Absolute Value**: - the absolute distance of a number from zero on the number line.




$$|\pm x| = x$$

**Solving Absolute Value Equations:**

1. **Isolate the Absolute Value Bracket on one side of the Equation.** ***Take off any Absolute Value Bracket*** **and** ***REWRITE the equation in TWO or MORE Scenarios*** **using a** ***POSITIVE Bracket*** **and a** ***NEGATIVE Bracket*****.**
2. ***SOLVE*** **for each scenario.**
3. ***VERIFY*** **the solutions by substituting them individually into the original equation.**
4. ***DISREGARD any EXTRANEOUS Solutions*** **(solutions that came about algebraically but do not work out when undergoing verification).**

**Example 1**: Solve the following equations. Verify all solutions.

a. $|2x - 3| = 9$

**Positive Case:**
$2x - 3 = +9$
$2x = 9 + 3$
$2x = 12$
$x = 6$

**Negative Case:**
$2x - 3 = -9$
$2x = -9 + 3$
$2x = -6$
$x = -3$

**Verify: For $x = 6$,**
$|2(6) - 3| = 9$
$|9| = 9$ ✓

**For $x = -3$,**
$|2(-3) - 3| = 9$
$|-9| = 9$ ✓

b. $5|(x - 2)| - 7 = 8$

$5|(x - 2)| - 7 = 8$
$5|(x - 2)| = 8 + 7$
$|(x - 2)| = \dfrac{15}{5}$
$|x - 2| = 3$

**Positive Case:**
$x - 2 = +3$
$x = 5$

**Negative Case:**
$x - 2 = -3$
$x = -1$

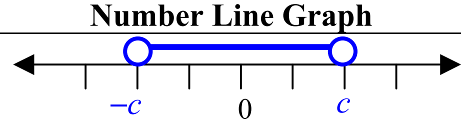
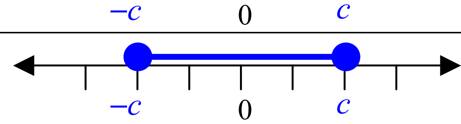
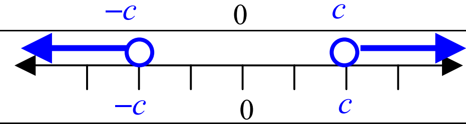
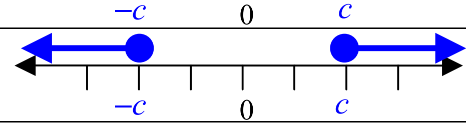
**Verify:**

**For $x = 5$,**
$5|(5) - 2| - 7 = 8$
$5|3| - 7 = 8$
$8 = 8$ ✓

**For $x = -1$,**
$5|(-1) - 2| - 7 = 8$
$5|-3| - 7 = 8$
$8 = 8$ ✓

**Solving Absolute Value Inequalities:**

1. **Isolate the Absolute Value Bracket on one side of the Equation.** ***Take off any Absolute Value Bracket*** **and** ***REWRITE the equation the Scenarios as specify in the following table*****.**
2. ***SOLVE*** **the simultaneous inequality or the separate union inequality.**
3. ***VERIFY*** **the solutions by substituting them individually into the original equation.**
4. ***DISREGARD any EXTRANEOUS Solutions*****.**

| Absolute Inequalities | Equivalent Inequality Forms | Number Line Graph |
|---|---|---|
| $\lvert x \rvert < c$ | $-c < x < c$ |   |
| $\lvert x \rvert \le c$ | $-c \le x \le c$ |   |
| $\lvert x \rvert > c$ | $x < -c$ or $x > c$ |   |
| $\lvert x \rvert \ge c$ | $x \le -c$ or $x \ge c$ |   |

**Example 2**: Show the absolute inequalities below follow their stated equivalent forms the previous table.

a. $|x| < c$ has solution $-c < x < c$

| Positive Case: | Negative Case: | |
|---|---|---|
| $+x < c$<br>$x < c$ | $-x < c$<br>$x > -c$ | Divide both sides by a negative. Need to switch inequality sign. |
| Hence, | $-c < x < c$ | |

b. $|x| \geq c$ has solution $x \leq -c$ or $x \geq c$

| Positive Case: | Negative Case: | |
|---|---|---|
| $+x \geq c$<br>$x \geq c$ | $-x \geq c$<br>$x \leq -c$ | Divide both sides by a negative. Need to switch inequality sign. |
| Hence, | $x \leq -c$ or $x \geq c$ | |

**Example 3**: Solve the following inequalities. Verify all solutions. Express the solutions as inequalities, graphs and in interval forms.

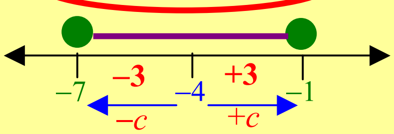
a. $|x + 4| \leq 3$

Using the Table on the last page,

$-3 \leq (x + 4) \leq +3$

$-3 - 4 \leq x \leq 3 - 4$

$-7 \leq x \leq -1$




$[-7, -1]$

*Note*: Midpoint between the two extremes is

$x = -4$

$(x + 4) = 0$ **(content of absolute value bracket)**

**Verify:** Let $x = -2$ (it is between −7 and −1)

$|(-2) + 4| \leq 3$

$|2| \leq 3$ $2 \leq 3$ ✓

b. $|3x - 1| > 11$

**Positive Case:**

$+(3x - 1) \geq 11$

$3x \geq 11 + 1$

$x \geq \frac{12}{3}$

$x \geq 4$

**Negative Case:**

$-(3x - 1) \geq 11$

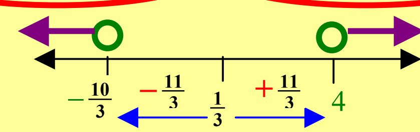
$-3x + 1 \geq 11$

$-3x \geq 11 - 1$

Divide by a negative. Need to switch inequality sign.

$x \leq -\frac{10}{3}$

$x \leq -\frac{10}{3}$ or $x \geq 4$ $\left(-\infty, \frac{10}{3}\right) \cup (4, \infty)$




**Verify:** Let $x = -3$ (less than $-\frac{10}{3}$)

$|3(-3) - 1| \geq 11$

$|-10| \geq 3$

$10 \geq 11$ ✓

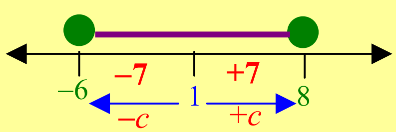
Let $x = 5$ (more than 4)

$|3(5) - 1| \geq 11$

$|14| \geq 11$

$14 \geq 11$ ✓

**Example 4**: State an absolute inequality that will give a solution of $-6 < x < 8$.




Midpoint between the two extremes is $\left(\frac{(-6)+(8)}{2}\right) = 1$.

$x = 1$

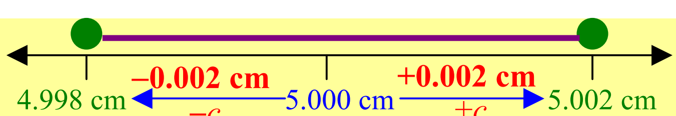
$(x - 1) = 0$ **(content of absolute value bracket)**

**Hence, $-6 \leq x \leq 8$ is a solution of** $|x - 1| < 7$

**Example 5**: A manufacturer of ball bearings has strict standards. One of its ball bearings produce has a diameter of 5.000 cm with a tolerance level of 0.002 cm. Write an absolute inequality that represents the diameter of this ball bearing. Solve the inequality for the range of diameters considered acceptable.

$d = 5.000$ cm (midpoint) or $(d - 5.000 \text{ cm}) = 0$

Tolerance = ± 0.002 cm




**Hence, 4.998 cm ≤ $d$ ≤ 5.002 cm is a solution of $|d - 5.000 \text{ cm}| < 0.002$ cm**

**1-7 Assignment: pg. 133–134 #5, 9, 13, 23, 27, 31, 37, 45, 47, 49, 51, 53;**
**Honours: #17 and 54**