10 A Square and Things

Quadratic Equations

The word "algebra" comes from a title of a book written in Arabic around the year 825. The author, Muḥammad Ibn Mūsa Al-Khwārizmī, was probably born in what is now Uzbekistan. He lived, however, in Baghdad, where the Caliph had established a kind of academy of science called "The House of Wisdom." Al-Khwārizmī was a generalist; he wrote books on geography, astronomy, and mathematics. But his book on algebra is one of his most famous.

Al-Khwārizmī's book starts out with a discussion of quadratic equations. In fact, he considers a specific problem:

> One square, and ten roots of the same, are equal to thirty-nine dirhems. That is to say, what must be the square which, when increased by ten of its own roots, amounts to thirty-nine?

If we call the unknown x, we might call the "square" x^2. Now, a "root of this square" is x, so "ten roots of the square" is $10x$. Using this notation, the problem translates into solving the equation $x^2 + 10x = 39$. But algebraic symbolism had not been invented yet, so all Al-Khwārizmī could do was to say it in words. In the time-honored tradition of algebra teachers everywhere, he follows the problem with a kind of recipe for its solution, again spelled out in words:

> The solution is this: you halve the number of the roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four. Now take the root of this, which is eight, and subtract from it half the number of the roots, which is five; the remainder is three. This is the root of the square which you sought for; the square itself is nine.[1]

Here's the computation in our symbols:

$$x = \sqrt{5^2 + 39} - 5 = \sqrt{25 + 39} - 5 = \sqrt{64} - 5 = 8 - 5 = 3.$$

It's not hard to see that this is basically just the quadratic formula as we now know it. To solve $x^2 + bx = c$, Al-Khwārizmī uses the rule

$$x = \sqrt{\left(\frac{b}{2}\right)^2 + c} - \frac{b}{2}.$$

The biggest difference between this and the modern formula is that we would consider both the positive and the negative square roots. But taking the negative square root would give a negative value for x. Mathematicians at the time didn't yet believe in negative numbers; the positive root was the only one they cared about. We also put the "$-b$" part at the beginning. But that would again mean a negative number, so he prefers to put it at the end, as a subtraction. (See Sketch 5.) Finally, he states the equation with the c on the right-hand side, while we would write it as $x^2 + bx - c = 0$.

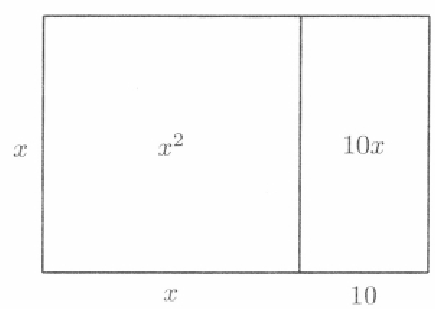
If we put the "$-b$" in front, add $\pm$ to the root, remember to take into account the sign of his c, and do a little algebra, his formula becomes ours:

$$x = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + c} = \frac{-b \pm \sqrt{b^2 + 4c}}{2}.$$

(The coefficient a is missing from this formula because Al-Khwārizmī was considering a single square; that is, $a = 1$.)

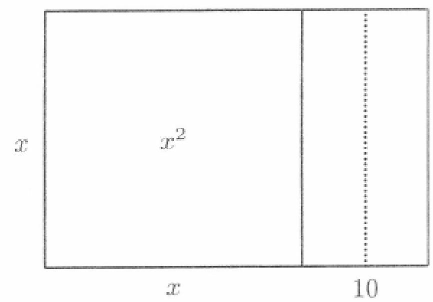
But he didn't leave it at that. He felt he should explain why his method worked. Rather than doing this algebraically, as we might today, he did it with a geometric argument. It went like this:

First, we have "a square and ten roots." To picture this, draw a square whose side we don't yet know. If we call the side x, the area of the square is x^2. To get $10x$, we draw a rectangle with one side equal to x and one side equal to 10, as in Display 1.



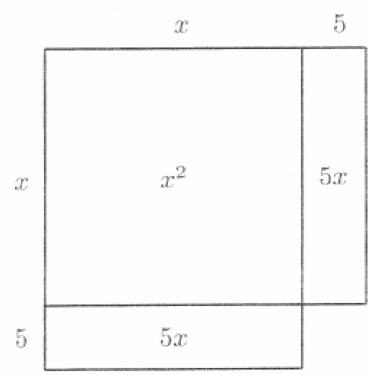
Display 1

The equation tells us that the area of the whole figure is 39. To solve the equation, that is, to determine x, we first cut the number of roots in half. Geometrically, that means we split the rectangle into two halves, each with area $5x$, as in Display 2.

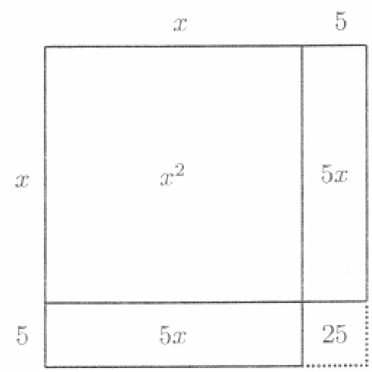


Display 2

Now we move one of the half-rectangles to the bottom of the square, making the figure in Display 3. The total area is still 39. But notice that adding in the little square missing in the lower right will make a big square. Since the two rectangles have side 5, the area of the little square must be 25, as shown in Display 4.



Display 3



Display 4

When we *complete the square* by adding in the little square, our figure becomes a square whose area is $39 + 25 = 64$. But this means that its side is equal to the square root of 64, which is 8. And since the side of the big square is $x + 5$, we can conclude that $x + 5 = 8$. So we subtract 5 and get $x = 3$.

Each step in Al-Khwārizmī's rule corresponds to a step in the geometric version. And the geometric version shows us exactly what is going on and why it works! As noted before, this version of the quadratic formula assumes that the leading coefficient is 1. Today, we write the general quadratic equation as $ax^2 + bx + c = 0$, allowing for a different leading coefficient. Al-Khwārizmī would have dealt with this simply by dividing through by a and applying his rule.

After Al-Khwārizmī's time many other mathematicians wrote about quadratic equations. Their methods and their geometric justifications became more and more sophisticated. But the basic idea never changed. In fact, even the example stayed the same. From the 9th century to the 16th century, books on algebra almost always started their discussion of quadratic equations by considering "a square and ten roots are equal to 39."

Early in the 17th century, mathematicians came up with the idea of using letters to represent numbers. (See Sketch 8.) Slowly, a convention developed: Letters from the end of the alphabet would denote *unknown* numbers, and letters from the beginning of the alphabet would stand for *known* numbers. Finally, Thomas Harriot and René Descartes noticed that it's much easier to write all equations as something = 0. The main advantage is that now $ax^2 + bx = c$ and $ax^2 + c = bx$ could be seen as special cases of the general equation $ax^2 + bx + c = 0$. And the general solution could now be written as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

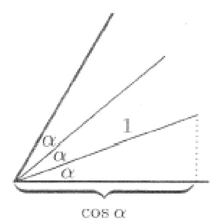
That's what we still do today.

11 Intrigue in Renaissance Italy Solving Cubic Equations

Mathematical problems rarely arise in abstract form. The problem of solving cubic equations (equations of degree three) grew out of geometric problems first considered by Greek mathematicians. The original problems may go back as far as 400 B.C., but the complete solution only came some 2000 years later.

The story begins with a famous geometric question: Given an angle, is there a way to construct an angle one third as large? To make sense out of this question, we first need to understand (or decide) what "construct" means. If it means using only a ruler and a compass, the answer is that it cannot be done. If we allow other tools, it can. Several constructions were known in Ancient Greece, many of them involving conic sections such as parabolas and hyperbolas.

Once trigonometry was developed, it became clear that this problem boils down to solving a cubic equation, as follows. To find one third of a given angle θ, we can begin by thinking of θ as three times the angle we're looking for, which we'll call α; that is, $\alpha = \theta/3$. Now we apply the formula for the cosine of 3α:



$$\cos(3\alpha) = 4\cos^3(\alpha) - 3\cos(\alpha).$$

Since the angle θ is known, we also know $\cos(\theta)$; call it a. To construct $\theta/3$, we need to construct its cosine. If we let $x = \cos(\theta/3)$, then, using the formula above with $\alpha = \theta/3$, we get $a = 4x^3 - 3x$, or $4x^3 - 3x - a = 0$. Finding x amounts to solving this equation.

When the Arabic mathematicians had begun doing algebra, it was inevitable that someone would try to apply the new techniques to equations of degree three. The most famous mathematician to attempt this was 'Umar Al-Khāyammī, known in the West as Omar Khayyam. Al-Khāyammī, who was born in Iran in 1048 and died in 1131, was famous in his time as a mathematician, scientist, and philosopher. He seems also to have been a poet, and that is how he is best known today.[1]

Because the Arabic mathematicians did not use negative numbers and did not allow zero as a coefficient, Al-Khāyammī had to consider

[1] His most famous poetic work is the Rubā'iāt, meaning "quatrains," which was (very freely) translated in 1859 by Edward FitzGerald as *The Rubáiyát of Omar Khayyám*.

many cases. For him, $x^3 + ax = b$ and $x^3 = ax + b$ were different kinds of equations. Arabic algebra was expressed entirely in words, so he described them as "a cube and roots are equal to a number" and "a cube is equal to roots and a number," respectively. Considered in this way, there are fourteen different kinds of cubic equations. For each of them, Al-Khāyammī found a geometric solution: a construction that yields a line segment whose length satisfies the equation. Most of these constructions involve intersecting conic sections, and many have side conditions to guarantee the existence of positive solutions.

Al-Khāyammī's work is impressive, but when it comes to determining a *number* that solves the equation it is of no help at all, as he himself acknowledges. That problem was left for others to attack.

Algebra reached Italy in the 13th century. Leonardo of Pisa's *Liber Abbaci* discussed both algebra and arithmetic with Hindu-Arabic numerals. In the following centuries, a lively tradition of arithmetic and algebra teaching developed in Italy. As Italian merchants developed their businesses, they had more and more need of calculation. The Italian "abbacists" tried to meet this need by writing books on arithmetic and algebra. Several of them discussed examples of cubic equations. In some cases, the examples were chosen so that the equations could be solved, or they were constructed from their solutions. In other cases, the authors presented incorrect ways to solve them. None had a complete solution of the general problem.

There was not much real progress on the problem until the work of Scipione del Ferro (1465–1526) and Niccolò Fontana (1500–1557), known as Tartaglia ("the Stammerer"). Both discovered how to solve certain cubics, and both kept their solutions secret. At this time, Italian scholars were mostly supported by rich patrons and had to prove their talent by defeating other scholars in public competitions. Knowing how to solve cubic equations allowed them to challenge others with problems that they knew the others could not solve. Thus, this competition system encouraged people to keep secrets.

In 1535 Tartaglia bragged that he could solve cubic equations, but he wouldn't tell anyone how he did it. Scipione del Ferro, who was dead by this time, had passed his own secret on to his student Antonio Maria Fiore. When Fiore heard of Tartaglia's claim, he challenged him to a competition. It turned out that del Ferro knew how to solve equations of the form $x^3 + cx = d$, and that Tartaglia had discovered how to solve $x^3 + bx^2 = d$. When the time for the contest came, Tartaglia presented Fiore with a range of questions on several different parts of mathematics, but each and every one of Fiore's questions boiled down to a cubic of the kind he could solve. Faced with this, Tartaglia managed to find a solution for this kind of equation, too, and won the contest handily when it turned out that Fiore's knowledge didn't extend much beyond cubic equations.

News of Tartaglia's victory eventually reached Girolamo Cardano (1501–1576), one of the most interesting figures of 16th century Italy. Cardano was a doctor, a philosopher, an astrologer, and a mathematician. In each of those fields he came to be well known and respected throughout Europe. In 1552, for example, he was invited to come to Scotland to help treat the Bishop of St. Andrews, who was suffering from serious asthma attacks. He agreed to go and was successful in curing the Bishop, and that solidified his fame.

Cardano's adventures with the cubic equation happened earlier in his life. Having heard of Tartaglia's solution, Cardano contacted him in 1539 to try to convince him to share the secret. Cardano's many pleas and promises of secrecy[2] eventually convinced Tartaglia, who came to Milan to explain his solution to Cardano. Once in possession of a method for solving a couple of cases of the cubic, Cardano attacked the problem of the general equation and, after six years of intense work, managed to solve it completely. His assistant, Lodovico Ferrari (1522–1565), applied the same set of ideas to the general equation of degree four (the *quartic*) and managed to find a solution for that, too.

At this point, Cardano knew that he had made a real contribution to mathematics. But how could he publish it without breaking his promise? He found a way. He discovered that del Ferro had found the solution of the crucial case before Tartaglia had. Since he had not promised to keep *del Ferro's* solution secret, he felt that he could publish it, even though it was identical to the one he had learned from Tartaglia. The resulting book was called *Ars Magna*, meaning "The Great Art," that is, algebra. It contains a complete account of how to solve any cubic equation, with geometric justifications of why the methods work. The book also includes Ferrari's solution of the quartic. Written in Latin, the book reached scholars all over Europe. And, of course, it reached Tartaglia.

Tartaglia was furious, but what could he do? The secret was out. He made public the story of Cardano's treachery, but Cardano was on to other things. Instead, Ferrari contacted Tartaglia and challenged him to a competition. Tartaglia considered Ferrari an unimportant

[2] According to Tartaglia, Cardano said "I swear to you, by God's holy Gospels, and as a true man of honour, not only never to publish your discoveries, if you teach me them, but I also promise you, and I pledge my faith as a true Christian, to note them down in code, so that after my death no one will be able to understand them." See [54], p. 255.

young man, so at first he was not interested in the challenge unless Cardano could be brought in, as well. But in 1548 Tartaglia was offered a professorship *on the condition that he defeat Ferrari in a contest.* He agreed, expecting to win easily. Ferrari, however, knew how to solve the general cubic and quartic equations, and Tartaglia had not absorbed that part of the *Ars Magna*. Tartaglia lost, and he remained resentful of Cardano to the end of his life.

This is not yet the end of the story, however. Applying Cardano's method to equations of the form $x^3 = px + q$, one sometimes ended up with expressions that didn't seem to make any sense. For example, for $x^3 = 15x + 4$, Cardano's method gives

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

Normally, one would conclude from the appearance of roots of negative numbers that the equation has no solution. But in this case the equation *does* have a solution, namely $x = 4$.

Cardano noticed this problem before he wrote the *Ars Magna*, and he asked Tartaglia about it. Tartaglia seems to have had no answer; he just suggested that Cardano had simply not understood how to solve such problems. It fell to Rafael Bombelli (1526–1572) to resolve the issue. Bombelli began by discussing the equation given above. He then showed, geometrically, that $x^3 = px + q$ always has a positive solution, regardless of the (positive) values of p and q. On the other hand, he showed that, for many values of p and q, solving this equation led to square roots of negative numbers. What Bombelli did at this point was nothing short of brilliant (for his time). He showed that it is possible to work with square roots of negative numbers and still get reasonable answers! (You can find more details about this in Sketch 17.)

With the cubic and quartic solved, the natural next target was the equation of degree 5. That proved to be a much more difficult problem. In fact, it turned out to be impossible to find a formula for solving the general quintic equation. Proving this required a complete change of point of view, which eventually led to the development of abstract algebra.

For a Closer Look: There is a good account of the solution of the cubic equation in [80, Chapter 9]. Al-Khāyammī's algebra has been translated into English; see [82]. Cardano's *Ars Magna* and his autobiography are also available in English as [26] and [27]. They offer a fascinating glimpse of the ways of thought of one of the most brilliant of Renaissance men.

After reading the two selections, please answer the following questions:

1. Why didn't Al-Khwarizmi's quadratic formula rule not give both positive and negative roots?

2. How would Al-Khwarizmi have solved $ax^2 + bx + c = 0$?

3. Describe the convention for using letters to represent numbers.

4. Which two Italian mathematicians dueled and what was the outcome?

5. How many different kinds of cubic equations are there according to Al-Khayammi?

6. Who was Cardano? What was his contribution to mathematical history?

7. What happened to Tartaglia?

8. Did anyone ever address the negative number issue? If so, who and whe